A MODEL FOR THE INEFFABLE TREE PROPERTY AT $\aleph_{\omega+2}$ AND STATIONARY REFLECTION AT $\aleph_{\omega+1}$

ALEJANDRO POVEDA AND DIMA SINAPOVA

Abstract. Assuming that $\langle \kappa_n \mid n < \omega\rangle^\frown\langle\lambda\rangle$ is an increasing sequence of supercompact cardinals we produce a model of ZFC where $\mathsf{GCH}_{<\aleph_\omega}$, $\mathrm{Refl}(\aleph_{\omega+1})$ and $\mathrm{ITP}(\aleph_{\omega+2})$ hold. This strengthens the main result of [PRS21]. This work gives a more decisive answer to a question of Magidor ([Mag77, Mag82]).

1. Introduction

Given a mathematical structure (a group, a graph, etc), suppose that all of its "small" substructures witness a property φ. Does the entire structure satisfy φ? A positive answer to this question is an instance of a phenomenon called *compactness*. Roughly speaking, compactness asserts that the local behaviour of a mathematical structure determines its nature.

Compactness usually holds when "small" means finite. For instance, a group G is free if and only if all of its finite subgroups $H \leq G$ are free. In the same spirit, a theorem of De Brujin and Erdös [BE51] establishes that a graph $\mathcal{G}$ has *chromatic number* $\leq n$ (for a fixed $n \in \mathbb{N}$) provided all of its finite subgraphs $\mathcal{H}$ have chromatic number $\leq n$. Perhaps the most famous example in this vein is Gödel's *Compactness Theorem* for first-order logic.

What if the small substructures are – rather than just finite – of size less than κ for a cardinal $\kappa \geq \aleph_1$? Should we still expect forms of compactness? In general, compactness fails at the level of $\aleph_1$; namely, when the substructures are countable. On the other hand, compactness can hold for higher cardinals, but this typically requires the existence of *large cardinals*. Large cardinal axioms are strengthenings of ZFC that assert the existence of certain "large" cardinals with strong compactness type properties.

This paper is concerned with two prominent set-theoretic manifestations of the compactness phenomenon – the *tree property* and *stationary reflection*. Given an uncountable regular cardinal κ the *tree property* holds at κ (in symbols, $\mathrm{TP}(\kappa)$) if every κ-tree T has a cofinal branch.[1] And *stationary reflection holds at* κ (in symbols, $\mathrm{Refl}(\kappa)$) if every stationary set $S \subseteq \kappa$

2020 *Mathematics Subject Classification.* Primary 03E55; Secondary 03E35.

Key words and phrases. Compactness, Singular cardinals, Prikry-type forcings.

[1] Recall that a κ-tree T is a tree with height κ and levels of cardinality $<\kappa$. For instance, the binary tree $2^{<\omega}$ is an $\aleph_0$-tree in that infinite and finitely-branching.

reflects; namely, there is $\alpha < \kappa$ of uncountable cofinality such that $S \cap \alpha$ is stationary in α.[2] Informally, this says that every "big enough" or "positive" subset of κ has big enough or positive initial segments.

The tree property at $\aleph_0$ is simply the conclusion of König's infinity lemma. In contrast, by Aronszajn, the tree property fails at the first uncountable cardinal, $\aleph_1$. Later, Mitchell proved in [Mit72] that $\mathrm{TP}(\aleph_2)$ is consistent (i.e., non contradictory with the ZFC axioms). He started with a type of large cardinal called *weakly compact* and used what is now called Mitchell forcing to turn it into $\aleph_2$. By Silver, this large cardinal hypothesis is necessary.

The tree property is closely tied with cardinals arithmetic. To get the tree property at $\aleph_2$, the *Continuum Hypothesis* (CH) must fail. More generally, Specker [Spe90] showed that the tree property at the double successor of a singular strong limit cardinal (e.g. $\aleph_{\omega+2}$) implies the failure of the *Singular Cardinal Hypothesis* (SCH). SCH is a parallel of CH for singular cardinals. The first result in this vein is due to Cummings and Foreman [CF98], who proved that $\mathrm{TP}(\kappa^{++})$ is consistent for κ singular strong limit.

The other property under consideration is stationary reflection. It is a classical result of Jensen that in Gödel's constructible universe $\mathrm{Refl}(\kappa)$ holds if and only if κ is weakly compact, affirming its large-cardinal strength. Regarding successors of singular cardinals, Magidor showed in [Mag82] that $\mathrm{Refl}(\aleph_{\omega+1})$ is consistent with ZFC. Recently, Hayut and Unger [HU20] reduced the large cardinal assumptions employed by Magidor.

An ongoing ambitious program in Set Theory is to determine to which extent various compactness principles can coexist. The focus of this paper is combining stationary reflection at the successor of a singular κ with the tree property at the double successor of κ.

The configuration "$\mathrm{Refl}(\aleph_{\omega+1})$ together with not SCH at $\aleph_\omega$" was a long-standing problem by Magidor, which was solved recently in the affirmative by the authors and A. Rinot ([PRS21]). Recall that, by Specker, when $\aleph_\omega$ is strong limit, $\mathrm{TP}(\aleph_{\omega+2})$ implies failure of SCH. Here we go further, by replacing "the failure of SCH at $\aleph_\omega$" by (the stronger) $\mathrm{TP}(\aleph_{\omega+2})$.

Main Theorem. *Assume the* GCH *holds. Suppose that* $\langle \kappa_n \mid n < \omega\rangle^\frown\langle\lambda\rangle$ *is an increasing sequence of supercompact cardinals. Then there is a generic extension of the set-theoretic universe where the following hold:*

(1) $\aleph_\omega = \sup_{n<\omega} \kappa_n$ *is strong limit and* $\lambda = \aleph_{\omega+2}$*;*
(2) $\mathsf{GCH}_{<\aleph_\omega}$ *holds but* $2^{\aleph_\omega} = \aleph_{\omega+2}$*;*
(3) $\mathrm{Refl}(\aleph_{\omega+1})$ *holds;*
(4) $\mathrm{TP}(\aleph_{\omega+2})$ *holds. Actually, the* Ineffable Tree Property (ITP) *holds.*

ITP is a stronger form of the tree property. It was isolated by Magidor [Mag74], and later studied by Weiß [Wei12]. Informally, just like the tree

[2] Recall that $S \subseteq \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for all $C \subseteq \kappa$ closed unbounded (*club*) with respect to the order topology of κ.

property captures the combinatorial essence of weak compactness, ITP captures supercompactness. More precisely, given an inaccessible cardinal κ, ITP holds at κ if and only if κ is supercompact.

The main theorem of this paper provides an even stronger positive response to Magidor's question. In [Git22] Gitik also demonstrates the compatibility of these principles at a cardinal high up. Obtaining the result for $\aleph_\omega$ requires different techniques. The reason is that putting collapses introduces nonreflecting stationary sets. Therefore, one needs an iteration to destroy these sets. As a starting point, we define a new Prikry-type forcing combining Mitchell forcing with the Gitik-Magidor *Extender-Based Prikry forcing* [GM94].

Let us now outline the content of the manuscript. In Section 2 we present our new forcing. In Section 3 we describe how to interleave collapses in it and show that the resulting forcing fits the $(\Sigma, \vec{\mathbb{S}})$-framework of [PRS21]. In Section 4 we present a variant of the Sharon's functor designed to kill nonreflecting stationary sets. Then, in Section 5, we describe the Σ-Prikry-styled forcing iteration $\mathbb{P}_\lambda$ which yields the model of our main theorem.

In Sections 6 and 7 we prove our main theorem. The key technical verification is that ITP($\aleph_{\omega+2}$) holds after forcing with $\mathbb{P}_\lambda$. We build upon previous works of Gitik [Git11] and Hatchman and Sinapova [HS19]. As a side result we obtain a self contained construction of a model with ITP at $\aleph_{\omega+2}$. Our proof is noticeably more involved than its ancestors as $\mathbb{P}_\lambda$ is way more complicated than the forcing used in [Git11, HS19].

Our readers are assumed to have some familiarity with Prikry-type forcings, in particular, with the Σ-Prikry iteration framework [PRS22, PRS21]. As in those papers, we use the following notation: Given a projection $\pi : \mathbb{Q} \to \mathbb{P}$, and conditions $q, r \in \mathbb{Q}$, we say that $q \leq^\pi r$ if $q \leq r$ and $\pi(q) = \pi(r)$. $\mathbb{Q}^\pi$ denotes the poset $(Q, \leq^\pi)$.

2. A Mitchell-like Extender Based Forcing

In this section we introduce a hybrid between the classical Mitchell forcing from [Mit72] and the Gitik-Magidor Extender Based Prikry forcing from [GM94]. The motivation is to devise a poset that forcess the tree property at the doule successor of a singular cardinal and has the nice properties of diagonal extender based forcing.

2.1. **Definition of $\mathbb{M}$.** Through this section $\Sigma := \langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of $(\lambda+1)$-strong cardinals such that λ is a regular cardinal with $\lambda > \kappa := \sup(\Sigma)$. For each $n < \omega$ we fix a $(\kappa_n, \lambda+1)$-extender E_n witnessing that κ_n is $(\lambda+1)$-strong. Let $\mathbb{P}$ be the Gitik-Magidor *Extender-based Prikry forcing* (EBPF) defined with respect to the sequence of extenders $\vec{E}$ [Git10, §2]. The present section is self-contained but familiarity with the notations in [Git10, §2] is assumed. $\mathcal{R}$ will denote the regular cardinals in $[\kappa^+, \lambda)$.

We will need a couple of technical observations about the EBPF and its projections. Given $p = \langle f_0^p, \dots, f_{\ell-1}^p, (a_\ell^p, A_\ell^p, f_\ell^p), \dots \rangle \in \mathbb{P}$ and $\alpha \leq \lambda$ such that every member of $\langle a_n^p \cap \alpha \mid n \geq \ell \rangle$ contains a $\leq_{E_n}$-maximal element, one can define its putative restriction $p \restriction \alpha$ to $\mathbb{P} \restriction \alpha$ as follows:[3]

$$p \restriction \alpha := \langle f_0^p \restriction \alpha, \dots, f_{\ell-1}^p \restriction \alpha, (a_\ell^p \cap \alpha, \pi_{\mathrm{mc}(a_\ell^p), \mathrm{mc}(a_\ell^p \cap \alpha)} \text{“} A_\ell^p, f_\ell^p \restriction \alpha), \dots \rangle.$$

Lemma 2.1 ([Git10, Lemma 2.2])**.** *Let $\theta \in \mathcal{R}$ and $n < \omega$. Then, the partial order $\leq_{E_n} \restriction (\theta \times \theta)$ is κ_n-directed. Moreover, for every $a \in [\theta]^{<\kappa_n}$ there are θ-many $\alpha < \theta$ such that $\alpha \geq_{E_n} \beta$ for all $\beta \in a$.*

If $\alpha \in \mathcal{R}$ and $p \in \mathbb{P}$ then one can use Lemma 2.1 to produce a condition $q \in \mathbb{P}$, $q \leq^* p$, with the same Cohen functions (i.e., $f_n^q = f_n^p$ for all $n < \omega$) and such that $q \restriction \alpha \in \mathbb{P} \restriction \alpha$. Pushing this idea further it is possible to isolate a $\leq^*$-dense subposet $\mathbb{Q}$ of $\mathbb{P}$ for which $p \restriction \alpha$ is always well-defined. Specifically, let us consider $\mathbb{Q}$ the subposet of $\mathbb{P}$ whose universe is

$$\{p \in \mathbb{P} \mid \forall n \geq \ell(p)\, \forall \alpha \in \mathcal{R}\ (\mathrm{mc}(a_n^p \cap \alpha) \text{ exists})\}.$$

Lemma 2.2. *$\mathbb{Q}$ is $\leq^*$-dense in $\mathbb{P}$.*

Proof. Let $p \in \mathbb{P}$, say, with $\ell(p) = 0$. Denote $F := \bigcup_{n<\omega} \mathrm{dom}(f_n)$, where

$$p = \langle (a_0, A_0, f_0) \dots (a_n, A_n, f_n), \dots \rangle.$$

By passing to a $\leq^*$-extension (if necessary) we can assume that $\kappa^+ \in a_0$ and that $\mathrm{mc}(a_0 \cap \kappa^+)$ exists (Lemma 2.1). We shall modify a_0 so as to ensure that $\mathrm{mc}(a_0 \cap \alpha)$ exists for all $\alpha \in \mathcal{R}$. For this, let $\langle \alpha_i \mid i < \sigma \rangle$ be the increasing enumeration of $a_0 \setminus \kappa^+$. For each $i < \sigma$, let $\theta_i \in [\alpha_i, \alpha_{i+1}) \cap \mathcal{R}$ be the first member of this intersection, if that exists. Otherwise, θ_i is not defined. This yields a sequence $\langle \theta_j \mid j < \sigma' \rangle$ of regular cardinals for some $\sigma' \leq \sigma < \kappa_0$. By induction on $j < \sigma'$ let us define $\langle b_j \mid j < \sigma' \rangle$ such that: $b_j \supseteq (a_0 \setminus \kappa^+)$, $b_j \subseteq b_{j+1}$ and $b_j \cap \theta_j$ has a $\leq_{E_0}$-maximal coordinate not in F. We can do that using Lemma 2.1 and $|F| < \theta_i$. Finally, let $c_0 := (a_0 \cap \kappa^+) \cup (\bigcup_{j<\sigma'} b_j) \cup \{\beta_{\sigma'}\}$ where $\beta_{\sigma'}$ is some ordinal below $\min(\mathcal{R} \setminus \sup(\bigcup_{j<\sigma'} b_j))$ that serves as a maximal coordinate and $\beta_{\sigma'} \notin F$.[4] Note that c_0 satisfies the required property; namely, $\mathrm{mc}(c_0 \cap \alpha)$ exists for every $\alpha \in \mathcal{R}$. To complete the proof let us recursively modify each a_n according to the above procedure. This yields a condition $p^* \leq^* p$ in $\mathbb{Q}$. $\square$

By virtue of Lemma 2.2, $\mathbb{Q}$ and $\mathbb{P}$ are forcing equivalent. As a result, we do not lose any generality by assuming that our EBPF poset is $\mathbb{Q}$. In a slight abuse of notation let us keep denoting this poset by $\mathbb{P}$.

The proof of the following lemma about projections is routine:

---

[3] This is a shorthand for $\mathbb{P}_{\vec{E}\restriction\alpha}$, the Extender-Based Prikry forcing defined with respect to the truncated sequence $\vec{E} \restriction \alpha := \langle E_n \restriction \alpha \mid n < \omega \rangle$.

[4] Note that since $|\bigcup_{j<\sigma'} b_j| < \kappa_0$ its supremum cannot be a regular cardinal.

Lemma 2.3. *Fix $\beta < \alpha$ in $\mathcal{R}$. Then,*

$$\cdot \restriction \alpha \colon p \mapsto p \restriction \alpha,$$

is a length-preserving projection from $\mathbb{P} \restriction \alpha$ to $\mathbb{P} \restriction \beta$.

These projections commute and, for each $p \in \mathbb{P}$ and $\vec{\nu} \in \prod_{\ell(p) \leq i \leq k} A^p_i$

$$(p^\frown \vec{\nu}) \restriction \alpha = (p \restriction \alpha)^\frown \langle \pi_{\mathrm{mc}(a^p_i), \mathrm{mc}(a^p_i \cap \alpha)}(\nu_i) \mid \ell(p) \leq i \leq k \rangle,$$

where $p^\frown \vec{\nu}$ is the weakest extension of p using the Prikry points given by $\vec{\nu}$.

With the above results at hand we can define the *Mitchell-like Extender Based Prikry forcing* (MEBPF) $\mathbb{M}$ as follows:

Definition 2.4. A condition in $\mathbb{M}$ is a pair (p, c) where $p \in \mathbb{P}$, c is a function with $\mathrm{dom}(c) \in [\mathcal{R}]^{\leq \kappa}$ and for each $\alpha \in \mathrm{dom}(c)$ the following hold:

1. $c(\alpha) \colon [\prod_{n \geq \ell(p)} \pi_{\mathrm{mc}(a^p_i), \mathrm{mc}(a^p_i \cap \alpha)} “A^p_i]^{<\omega} \to V^{\mathbb{P} \restriction \alpha}$;[5]
2. $\mathbb{1} \Vdash_{\mathbb{P} \restriction \alpha}$ “$c(\alpha, \vec{\pi}) \in \dot{\mathrm{A}}\mathrm{dd}(\kappa^+, 1)$”, for all $\vec{\pi} \in \mathrm{dom}(c(\alpha))$;
3. $(p \restriction \alpha)^\frown \vec{\sigma} \Vdash_{\mathbb{P} \restriction \alpha} c(\alpha, \vec{\sigma}) \leq c(\alpha, \vec{\pi})$, for all $\vec{\pi} \sqsubseteq \vec{\sigma}$ in $\mathrm{dom}(c(\alpha))$.

Given $(p, c), (q, d) \in \mathbb{M}$ we write $(p, c) \leq^* (q, d)$ if and only if:

(I) $p \leq^* q$;
(II) $\mathrm{dom}(c) \supseteq \mathrm{dom}(d)$;
(III) for $\alpha \in \mathrm{dom}(d)$ and $\vec{\pi} = \langle \pi_\ell, \dots, \pi_k \rangle \in \prod_{\ell \leq i \leq k} \pi_{\mathrm{mc}(a^p_i), \mathrm{mc}(a^p_i \cap \alpha)} “A^p_i$,

$$(p \restriction \alpha)^\frown \vec{\pi} \Vdash_{\mathbb{P} \restriction \alpha} “c(\alpha, \vec{\pi}) \leq d(\alpha, \langle \pi_{\mathrm{mc}(a^p_i \cap \alpha), \mathrm{mc}(a^q_i \cap \alpha)}(\pi_i) \mid \ell \leq i \leq k \rangle)”.$$

Remark 2.5. It is routine to check that clause (III) above is well-defined. This is a consequence of $p \leq^* q$ and the commutativity of the projections associated to the EBPF. The idea behind $\mathbb{M}$ is the following. As usual, the EBPF-*part* (incarnated by p) contributes to generating an initial segment $\vec{\nu} := \langle \nu_{\ell(p)}, \dots \nu_k \rangle$ of one of the eventual EBPF-generics. This $\vec{\nu}$ guides the *Cohen part* (represented by c) to collapse all V-regular cardinals in (κ^+, λ).

Let us now define the n-point extensions:

Definition 2.6. Let $(p, c) \in \mathbb{M}$ and $\nu \in A^p_\ell$. Denote $(p, c)^\frown \nu := (p^\frown \nu, c^\frown \nu)$ where $c^\frown \nu$ is the function defined as follows:

1. $\mathrm{dom}(c^\frown \nu) := \mathrm{dom}(c)$;
2. $\mathrm{dom}((c^\frown \nu)(\alpha))$ is the collection of all $\langle \pi_{\ell+1}, \dots, \pi_k \rangle$ such that
$$\langle \pi_{\mathrm{mc}(a^p_\ell), \mathrm{mc}(a^p_\ell \cap \alpha)}(\nu) \rangle^\frown \langle \pi_{\ell+1}, \dots, \pi_k \rangle \in \mathrm{dom}(c(\alpha));$$
3. for each $\vec{\pi} = \langle \pi_{\ell+1}, \dots, \pi_k \rangle \in \mathrm{dom}((c^\frown \nu)(\alpha))$,
$$(c^\frown \nu)(\alpha, \vec{\pi}) := c(\alpha, \langle \pi_{\mathrm{mc}(a^p_i), \mathrm{mc}(a^p_i \cap \alpha)}(\nu) \rangle^\frown \langle \pi_{\ell+1}, \dots, \pi_k \rangle).$$

In general, one defines $(p, c)^\frown \vec{\nu}$ by recursion on the length of $\vec{\nu}$. More explicitly, $(p, c)^\frown \vec{\nu} := (p^\frown \vec{\nu}, c^\frown \vec{\nu})$ where for each $\langle \pi_{|\vec{\nu}|+1}, \dots, \pi_k \rangle$,

$$(c^\frown \vec{\nu})(\alpha, \vec{\pi}) := c(\alpha, \langle \pi_{\mathrm{mc}(a^p_i), \mathrm{mc}(a^p_i \cap \alpha)}(\nu_i) \mid \ell \leq i \leq |\vec{\nu}| \rangle^\frown \langle \pi_{|\vec{\nu}|+1}, \dots, \pi_k \rangle).$$

[5]More verbosely, $\mathrm{dom}(c(\alpha))$ consists of finite sequences $\vec{\pi} = \langle \pi_{\ell(p)}, \dots, \pi_k \rangle$ in the product of the measure one sets of p and its outcome $c(\alpha, \vec{\pi})$ is a $\mathbb{P} \restriction \alpha$-name.

Remark 2.7. Roughly speaking, $c^\frown\nu$ is the *shift of* c by ν. Namely, $c^\frown\nu$ is the natural restriction of c where the ℓ-entry is fixed to be the projections of the Prikry point $\vec{\nu}$. In general, $c^\frown\vec{\nu}$ is the natural restriction of c where the first $|\vec{\nu}|$-many coordinates are fixed to be projections of $\vec{\nu}$. Using Lemma 2.3 and the definition of $\mathbb{M}$ it is routine to check that $(p,c)^\frown\vec{\nu}$ is a condition.

Finally, let us define the order $\leq$ of $\mathbb{M}$:

Definition 2.8. Given $(p,c),(q,d)\in\mathbb{M}$ let us write $(p,c)\leq(q,d)$ if there is a sequence $\vec{\nu}\in\prod_{\ell(q)\leq i\leq\ell(p)}A^q_i$ such that $(p,c)\leq^*(q,d)^\frown\vec{\nu}$.

Next, we prove the main properties of $\mathbb{M}$. We will show that $\mathbb{M}$ has the *Complete Prikry Property* (see [PRS22, §2]), analyze the cardinal structure, and finally prove that $\mathbb{M}$ projects to $\mathbb{M}\restriction\alpha$.[6]

2.2. **Prikry property.** For the next lemma, recall the definition of property $\mathcal{D}$ from [PRS22, Definition 2.12].

Lemma 2.9. $\mathbb{M}$ *has property* $\mathcal{D}$.

Proof. Let $a=(p,c)\in\mathbb{M}$, $n<\omega$ and $\vec{r}=\langle\vec{\nu}_\alpha\mid\alpha<\kappa_{\ell(p)+n-1}\rangle$ be an injective enumeration of the product of the first n-many measure one sets of p. Our aim is to show that **I** has a winning strategy in the diagonalizability game $\partial_{\mathbb{M}}(a,\vec{r})$. For simplicity, let us provide details just when $n=1$. Write $\ell:=\ell(p)$. At the beginning, **I** plays $a_0:=a$ and, in response, **II** plays some $(q_0,d_0):=b_0\leq^* {a_0}^\frown\vec{\nu}_0$. In the next round, **I** plays a pair $a_1:=(p_1,c_1)\leq^* a_0$ attempting to *diagonalize* b_0. More explicitly,

$$p_1:=\langle f^{q_0}_0,\dots,f^{q_0}_{\ell-1},(a^p_\ell,A^p_\ell,f^{q_0}_\ell\setminus a^p_\ell)),(a^{q_0}_{\ell+1},A^{q_0}_{\ell+1},f^{q_0}_{\ell+1}),\dots\rangle,$$

and c_1 is a function with

- $\mathrm{dom}(c_1):=\mathrm{dom}(d_0)$;
- $\mathrm{dom}(c_1(\alpha)):=[\prod_{i\geq\ell}\pi_{\mathrm{mc}(a^{p_1}_i),\mathrm{mc}(a^{p_1}_i\cap\alpha)}\text{“}A^{p_1}_i]^{<\omega}$,
- Fix $\vec{\pi}=\langle\pi_\ell,\cdots,\pi_k\rangle\in\mathrm{dom}(c_1(\alpha))$.

If $|\vec{\pi}|\geq 2$ and $\pi_\ell=\pi_{\mathrm{mc}(a^p_\ell),\mathrm{mc}(a^p_\ell\cap\alpha)}(\nu_0)$, set:

$$c_1(\alpha,\vec{\pi}):=d_0(\alpha,\langle\pi_{\ell+1},\cdots,\pi_k\rangle).$$

Otherwise,

$$c_1(\alpha,\vec{\pi}):=\begin{cases}c_0(\alpha,\langle\pi_{\mathrm{mc}(a^{p_1}_i\cap\alpha),\mathrm{mc}(a^p_i\cap\alpha))}(\pi_i)\mid i\in[\ell,k]\rangle), & \text{if }\alpha\in\mathrm{dom}(c_0);\\ \{(\check{\varnothing},\mathbb{1}_{\mathbb{P}\restriction\alpha})\}, & \text{otherwise.}\end{cases}$$

Claim 2.9.1. $(p_1,c_1)\in\mathbb{M}$, $(p_1,c_1)\leq^*(p_0,c_0)$ *and* $(p_1,c_1)^\frown\nu_0=b_0$.

Proof of claim. Clearly, $(p_1,c_1)^\frown\nu_0=b_0$. To see that $(p_1,c_1)\in\mathbb{M}$, it suffices to check that c_1 is well-defined and increasing when more Prikry points are chosen (see (2) and (3) of Definition 2.4). Clause (2) is fairly easy to verify. As for Clause (3) we argue as follows: Fix $\vec{\pi}\sqsubset\vec{\sigma}$ in $\mathrm{dom}(c_1(\alpha))$.

[6]In analogy with $\mathbb{P}\restriction\alpha$, $\mathbb{M}\restriction\alpha$ denotes the poset defined as $\mathbb{M}$ but using $\mathbb{P}\restriction\alpha$ as EBPF-part members and Cohen-parts c restricted to ordinals in $\mathcal{R}\cap\alpha$.

Case $\alpha \in \mathrm{dom}(d_0) \setminus \mathrm{dom}(c_0)$: If $\vec{\pi}$ witnesses the property of the first case then so does $\vec{\sigma}$. In that scenario, $(p_1 \restriction \alpha)^\frown \vec{\pi} = (q_0 \restriction \alpha)^\frown \langle \pi_{\ell+1}, \dots, \pi_k \rangle$ (and similarly for $\vec{\sigma}$). Thus, $(p_1 \restriction \alpha)^\frown \vec{\sigma} \Vdash_{\mathbb{P}\restriction\alpha}$ "$d_0(\alpha, \vec{\sigma} \setminus \ell) \leq d_0(\alpha, \vec{\pi} \setminus \ell)$" because (q_0, d_0) is a condition. In case $|\vec{\pi}| = 1$ but $\pi_\ell = \pi_{\mathrm{mc}(a^p_\ell),\mathrm{mc}(a^p_\ell \cap \alpha)}(\nu_0)$ it follows that $(p_1 \restriction \alpha)^\frown \vec{\sigma} = (q_0 \restriction \alpha)^\frown \langle \sigma_{\ell+1}, \dots, \sigma_k \rangle$ and clearly this forces $d_0(\alpha, \langle \sigma_{\ell+1}, \dots, \sigma_k \rangle) \leq \{\langle \check{\varnothing}, \mathbb{1}_{\mathbb{P}\restriction\alpha} \rangle\}$. In other case we fall into the second alternative and the desired property holds.

Case $\alpha \in \mathrm{dom}(c_0)$: If $\vec{\pi}$ is as in the first alternative one can argue as before that $(p_1 \restriction \alpha)^\frown \sigma$ forces $c_1(\alpha, \vec{\sigma}) \leq c_1(\alpha, \vec{\pi})$. Also, if $\pi_\ell \neq \pi_{\mathrm{mc}(a^p_\ell),\mathrm{mc}(a^p_\ell \cap \alpha)}(\nu_0)$ then both $\vec{\pi}$ and $\vec{\sigma}$ fall into the second alternative and one can use that $(p_1 \restriction \alpha)^\frown \vec{\sigma} \leq (p_0 \restriction \alpha)^\frown \langle \pi_{\mathrm{mc}(a^{p_1}_i \cap \alpha),\mathrm{mc}(a^p_i \cap \alpha)}(\sigma_i) \mid \ell \leq i \leq k \rangle$ and that $(p_0, c_0) \in \mathbb{M}$ to infer the desired property. So, we are left with the case where $|\vec{\pi}| = 1$ and $\pi_\ell = \pi_{\mathrm{mc}(a^p_\ell),\mathrm{mc}(a^p_\ell \cap \alpha)}(\nu_0)$. In that scenario

$$c_1(\alpha, \pi_\ell) = c_0(\alpha, \pi_\ell) \text{ and } c_1(\alpha, \vec{\sigma}) = d_0(\alpha, \langle \sigma_{\ell+1}, \dots, \sigma_{k+1} \rangle).$$

The proof will be completed once we check that

$$(\star) \qquad (p_1 \restriction \alpha)^\frown \vec{\sigma} \Vdash_{\mathbb{P}\restriction\alpha} d_0(\alpha, \langle \sigma_{\ell+1}, \dots, \sigma_k \rangle) \leq c_0(\alpha, \pi_\ell).$$

Recall that $(q_0, d_0) \leq^* (p_0, c_0)^\frown \nu_0$, which by definition implies

$$(q_0 \restriction \alpha)^\frown (\vec{\sigma} \setminus \ell) \Vdash_{\mathbb{P}\restriction\alpha} d_0(\alpha, \vec{\sigma} \setminus \ell) \leq c_0(\alpha, \langle \pi_\ell \rangle^\frown \langle \pi_{\mathrm{mc}(a^{q_0}_i \cap \alpha),\mathrm{mc}(a^p_i \cap \alpha)}(\sigma_i) \mid i \geq \ell+1 \rangle).^7$$

Since $(q_0 \restriction \alpha)^\frown (\vec{\sigma} \setminus \ell) \leq (p_0 \restriction \alpha)^\frown \langle \pi_{\mathrm{mc}(a^{p_1}_i \cap \alpha),\mathrm{mc}(a^p_i \cap \alpha)}(\sigma_i) \mid i \geq \ell + 1 \rangle$,

$$(q_0 \restriction \alpha)^\frown (\vec{\sigma} \setminus \ell) \Vdash_{\mathbb{P}\restriction\alpha} c_0(\alpha, \langle \pi_\ell \rangle^\frown \langle \pi_{\mathrm{mc}(a^{q_0}_i \cap \alpha),\mathrm{mc}(a^p_i \cap \alpha)}(\sigma_i) \mid i \geq \ell+1 \rangle) \leq c_0(\alpha, \pi_\ell)$$

because (p_0, c_0) was a condition. All in all, as $(p_1 \restriction \alpha)^\frown \vec{\sigma} = (q_0 \restriction \alpha)^\frown (\vec{\sigma} \setminus \ell)$, the last two expressions together imply that $(\star)$ holds. The above arguments also show that $(p_1, c_1) \leq^* (p_0, c_0)$. $\square$

After defining $a_1 := (p_1, c_1)$, **II** plays in response $b_1 \leq^* (p_1, c_1)^\frown \nu_1$. In general, suppose that $\langle (a_\xi, b_\xi) \mid \xi < \zeta \rangle$, $\zeta \leq \kappa_l$, has been formed according to the rules of $\eth_{\mathbb{M}}(a, \vec{r})$. If $\zeta = \xi + 1$ then let **I** play $a_\zeta = (p_\zeta, c_\zeta)$ where

$$p_\zeta := \langle f^{q_\xi}_0, \dots, f^{q_\xi}_{\ell-1}, (a^p_\ell, A^p_\ell, f^{q_\xi}_\ell \setminus a^p_\ell)), (a^{q_\xi}_{\ell+1}, A^{q_\xi}_{\ell+1}, f^{q_\xi}_{\ell+1}), \dots \rangle,$$

and c_ζ is defined as c_1 but using (a_ξ, b_ξ) instead of (a_0, b_0). Arguing as in the previous claim one shows that $a_\zeta := (p_\zeta, c_\zeta) \in \mathbb{M}$ and that $a_\zeta{}^\frown \nu_\xi = b_\xi$.

Otherwise,

$$p_\zeta := \langle \textstyle\bigcup_{\xi<\zeta} f^{q_\xi}_0, \dots, \bigcup_{\xi<\zeta} f^{q_\xi}_{\ell-1}, (a^p_\ell, A^p_\ell, (\bigcup_{\xi<\zeta} f^{q_\xi}_\ell) \setminus a^p_\ell)), (a^{p_\zeta}_{\ell+1}, A^{p_\zeta}_{\ell+1}, f^{p_\zeta}_{\ell+1}), \dots \rangle,$$

where the $(a^{p_\zeta}_{\ell+1}, A^{p_\zeta}_{\ell+1}, f^{p_\zeta}_{\ell+1})$'s are the result of taking $\leq^*$-lower bounds on the sequence $\langle q_\xi \setminus \ell \mid \zeta < \xi \rangle$.[8] Next, define c_ζ as the function with:

- $\mathrm{dom}(c_\zeta) := \bigcup_{\xi<\zeta} \mathrm{dom}(c_\xi)$,
- $c_\zeta(\alpha) := [\prod_{i \geq \ell} \pi_{\mathrm{mc}(a^{p_\zeta}_i),\mathrm{mc}(a^{p_\zeta}_i \cap \alpha)} {}^{``}A^{p_\zeta}_i]^{<\omega}$,

[7] Note that here we used $\pi_{\mathrm{mc}(a^p_\ell),\mathrm{mc}(a^p_\ell \cap \alpha)}(\nu_0) = \pi_\ell$.

[8] Such lower bounds exists because $\zeta < \kappa_{\ell+1}$.

Let $\alpha \in \mathrm{dom}(c_\xi)$ and denote by Λ_α the collection of all $\xi < \zeta$ such that $\alpha \in \mathrm{dom}(c_\xi)$. Let $\vec{\pi} = \langle \pi_\ell, \ldots, \pi_k \rangle$ in $\mathrm{dom}(c_\zeta(\alpha))$. To ease notations put

$$\vec{\pi}_{\xi,\alpha} := \langle \pi_{\mathrm{mc}(a_i^{p_\zeta} \cap \alpha), \mathrm{mc}(a_i^{p_\xi} \cap \alpha)}(\pi_i) \mid \ell \leq i \leq k \rangle.$$

For each $\xi < \eta$ in Λ_α we have, by construction,

$$(p_\zeta \restriction \alpha)^\frown \vec{\pi} \leq^* (p_\eta \restriction \alpha)^\frown \vec{\pi}_{\eta,\alpha} \leq^* (p_\xi \restriction \alpha)^\frown \vec{\pi}_{\xi,\alpha}$$

and

$$(p_\eta \restriction \alpha)^\frown \vec{\pi}_{\eta,\alpha} \Vdash_{\mathbb{P}\restriction\alpha} \text{“}\langle c_\xi(\alpha, \vec{\pi}_{\xi,\alpha}) \mid \xi \in \Lambda_\alpha \cap \eta \rangle \text{ is } \leq_{\dot{\mathrm{Add}}(\kappa^+,1)}\text{-decreasing”}.$$

So, let $c_\zeta(\alpha, \vec{\pi})$ is a $\mathbb{P} \restriction \alpha$-name for

$$\bigwedge \{ c_\xi(\alpha, \vec{\pi}_{\xi,\alpha}) \mid \xi \in \Lambda_\alpha \};$$

namely, a lower bound for the displayed sequence, as forced by $(p_\zeta \restriction \alpha)^\frown \vec{\pi}$.

Then, we have that

$$(p_\zeta \restriction \alpha)^\frown \vec{\pi} \Vdash_{\mathbb{P}\restriction\alpha} \forall \xi \in \Lambda_\alpha \, (c_\zeta(\alpha, \vec{\pi}) \leq_{\dot{\mathrm{Add}}(\kappa^+,1)} c_\xi(\alpha, \vec{\pi}_{\xi,\alpha})),$$

By tweaking $c_\zeta(\alpha, \vec{\pi})$ a bit we can moreover assume that it is a $\mathbb{P}\restriction\alpha$-name for a member of $\dot{\mathrm{Add}}(\kappa^+, 1)$ as forced by the trivial condition. In a similar spirit, given $\vec{\pi} \sqsubseteq \vec{\sigma}$ in $\mathrm{dom}(c_\zeta(\alpha))$, one can argue that

$$(p_\zeta \restriction \alpha)^\frown \vec{\sigma} \Vdash_{\mathbb{P}\restriction\alpha} c_\zeta(\alpha, \vec{\sigma}) \leq c_\xi(\alpha, \vec{\pi}_{\xi,\alpha}).$$

Claim 2.9.2. *$(p_\zeta, c_\zeta) \in \mathbb{M}$, $(p_\zeta, c_\zeta) \leq^* (p_\xi, c_\xi)$ and $(p_\zeta, c_\zeta)^\frown \nu_\xi \leq^* b_\xi$.*

Proof of claim. The first claim is obvious attending to our previous considerations. The claim that $(p_\zeta, c_\zeta)^\frown \nu_\xi \leq^* b_\xi$ follows from our inductive assumption; in effect, $(p_\zeta, c_\zeta)^\frown \nu_\xi \leq^* (p_{\xi+1}, c_{\xi+1})^\frown \nu_\xi \leq^* b_\xi$. □

The above defines the ζth-move of **I**, $a_\zeta := (p_\zeta, c_\zeta)$ for $\zeta \leq \kappa_l$. Letting **II** play some $b_\zeta \leq^* a_\zeta{}^\frown \nu_\zeta$ for $\zeta < \kappa_l$ yields a sequence $\langle (a_\zeta, b_\zeta) \mid \zeta < \kappa_\ell \rangle$.

To show that $\langle b_\zeta \mid \zeta < \kappa_\ell \rangle$ is diagonalizable, let $a' := (p_{\kappa_\ell}, c_{\kappa_\ell})$. Then $a' \leq^* a$ and $a'^\frown \nu_\zeta \leq^* a_{\zeta+1}{}^\frown \nu_\zeta = b_\zeta$ for all $\zeta < \kappa_\ell$. This shows that **I** has a winning strategy in $\Game_{\mathbb{M}}(a, \vec{r})$, as wished. □

Lemma 2.10. *$\mathbb{M}$ has the Complete Prikry Property.*

Proof. Fix $a = (p, c) \in \mathbb{M}$, $n < \omega$ and U a $\leq^*$-open subset of $\mathbb{M}$, i.e. if $b \in U$ and $b' \leq^* b$, then $b' \in U$. We will find $a' \leq^* a$ with the following property: either every $b \leq a$ with length $\ell(a) + n$ is in U or all of them avoid U. For simplicity, assume $\ell(a) = 0$. The general case is analogous.

Claim 2.10.1. *There is $a_0 \leq^* a$ such that, for every $b \leq a_0$ with $\ell(b) = n$, if $b \in U$ then $a_0{}^\frown \vec{\nu} \in U$, where $\vec{\nu}$ is the unique such that $b \leq^* a_0{}^\frown \vec{\nu}$.*

Proof of claim. Let $\vec{r} = \langle \vec{\nu}_\xi \mid \xi < \kappa_n \rangle$ be an injective enumeration of $\prod_{i \leq n} A_i^p$ (i.e., of the n-point extensions of a). Let us play the game $\Game_{\mathbb{M}}(a, \vec{r})$ but this time restricting the moves of **II** to U when this choice is possible. Specifically, let us define by induction a sequence $\langle (a_\xi, b_\xi) \mid \xi < \kappa_n \rangle \subseteq \mathbb{M}$ with $a_\xi \leq^* a$ and $b_\xi \leq^* a_\xi{}^\frown \vec{\nu}_\xi$ and if there is $b \leq^* a_\xi{}^\frown \vec{\nu}_\xi$ in U then **II** picks such

$b_\xi \in U$; otherwise, **II** chooses $b_\xi := a_\xi{}^\frown\vec{\nu}_\xi$. By Lemma 2.9 there is $a_0 \leq^* a$ diagonalizing $\langle b_\xi \mid \xi < \kappa_n\rangle$.

We claim that a_0 is as desired.[9] Let $b \leq a_0$ in U with $\ell(b) = n$. By definition, there is $\vec{\nu}_\xi$ such that $b \leq^* a_0{}^\frown\vec{\nu}_\xi$. Since a_0 diagonalizes the b_ξ's we have that $b \leq^* a_\xi{}^\frown\vec{\nu}_\xi$. In particular, b_ξ was chosen to be in U. Since $a_0{}^\frown\vec{\nu}_\xi \leq^* b_\xi \in U$ it follows (by $\leq^*$-openness) that $a_0{}^\frown\vec{\nu}_\xi \in U$, as well. □

Let us now move to the *Röwbottom part* of the argument for the CPP. Namely, we define $b \leq^* a$ such that $b = (q, d)$ and

$$\text{“}\forall\vec{\nu} \in \prod_{i<n} A^q_i\ (b^\frown\vec{\nu} \in U)\text{''or “}\forall\vec{\nu} \in \prod_{i<n} A^q_i\ (b^\frown\vec{\nu} \notin U)\text{''}.$$

We just give details of the construction when $n = 2$, which suffices to understand the general case.

For each $\nu \in A^a_0$ define

$$B^0_\nu := \{\eta \in A^{p_0}_1 \mid a^\frown\langle\eta,\nu\rangle \in U\} \text{ and } B^1_\nu := A^{p_0}_1 \setminus B^0_\nu.$$

For each $\nu \in A^a_0$ let B_ν be the unique of the above two sets which is $E_{\mathrm{mc}(a^{p_0}_1)}$-large. Put $B^0 := \{\nu \in A^a_0 \mid B_\nu = B^0_\nu\}$ and $B^1 := \{\nu \in A^a_0 \mid B_\nu = B^1_\nu\}$.

Let A_0 be the unique of the above which is $E_{\mathrm{mc}(a^{p_0}_0)}$-large and define $A_1 := \bigcap_{\nu\in A^{a_2}_0} B_\nu$, which is $E_{\mathrm{mc}(a^{p_0}_1)}$-large by completeness.

Define q as p but replacing the first two measure one sets by A_0 and A_1, respectively. As before, define d with the same domain and values as c_0 but the $\vec{\pi} \in \mathrm{dom}(d(\alpha))$'s are taken from the αth-projection of the measure one sets of q. □

Combining the CPP of $\mathbb{M}$ (Lemma 2.10) with the following easy lemma we infer that $\mathbb{M}$ has both the *Strong Prikry Property* and the *Prikry Property.*

Lemma 2.11. *For each $n < \omega$, the poset*

$$\mathbb{M}_n := \{(p, c) \in \mathbb{M} \mid \ell(p) = n\}$$

is κ_n-directed-closed.

Proof. Let $D \subseteq \mathbb{M}_n$ be a directed subset of conditions with $|D| < \kappa_n$. Let q be a $\leq^*$-lower bound for the EBPF-part of every $(p, c) \in D$. Let c^* be the function defined similarly to c_ζ in page 7. Namely, the domain of c^* is $\bigcup_{(p,c)\in D} \mathrm{dom}(c)$ and for each $\alpha, \vec{\pi}$, $c^*(\alpha, \vec{\pi})$ is obtained by taking lower bounds of relevant $c(\alpha, \vec{\sigma})$, where $(p, c) \in D$. As argued in Claim 2.9.2, $(q, c^*) \in \mathbb{M}$. Clearly it defines a $\leq^*$-lower bound for the conditions in D. □

Corollary 2.12.

(1) $\mathbb{M}$ has the Prikry Property; namely, for every sentence φ in the forcing language of $\mathbb{M}$ and every $a \in \mathbb{M}$ there is $b \leq^* a$ deciding φ.

[9] It is important for the forthcoming argument that the first n-many measure one sets appearing in a_0 are exactly $\prod_{i\leq n} A^p_i$. This is exactly what the proof of Lemma 2.9 shows.

(2) $\mathbb{M}$ *has the Strong Prikry Property; namely, for every* $a \in \mathbb{M}$ *and* $D \subseteq \mathbb{M}$ *dense open there is* $n < \omega$ *and* $b \leq^* a$ *such that every* $c \leq b$ *with* $\ell(c) \geq \ell(b) + n$ *is in* D.

2.3. **Cardinal structure.** Let us now analyze the cardinal structure of $V^{\mathbb{M}}$.

Lemma 2.13. *Forcing with* $\mathbb{M}$ *preserves all cardinals* $\leq \kappa^+$.

Proof. The preservation of cardinals $\leq \kappa$ follows from Corollary 2.12(1) and Lemma 2.11. Similarly, the prerservation of κ^+ can be established using Corollary 2.12(2) and the fact that $|\{(p,c)^\frown \vec{\nu} \mid \vec{\nu} \in [\prod_{n \geq \ell(p)} A^p_n]^{<\omega}\}| \leq \kappa$ for all $(p,c) \in \mathbb{M}$. The argument in both cases is standard, but further details can be found in [PRS19, Lemma 2.10]. □

Lemma 2.14. *Forcing with* $\mathbb{M}$ *collapses all* V*-regular cardinals in* (κ^+, λ).

Proof. Let $\alpha \in (\kappa^+, \lambda)$ be a V-regular cardinal and G a $\mathbb{M}$-generic filter over V. Since $\mathbb{M}$ projects to $\mathbb{P}$ and this latter to $\mathbb{P} \restriction \alpha$ we can respectively derive $\mathbb{P}$ and $\mathbb{P} \restriction \alpha$-generics $\bar{G}$ and $\bar{G}_\alpha$. Working in $V[G]$ the putative αth-collapsing function is defined as follows:

$$c_\alpha := \bigcup \{c(\alpha, \vec{\nu} \restriction \alpha)_{\bar{G}_\alpha} \mid (p,c) \in G,\ \alpha \in \mathrm{dom}(c),\ p^\frown \vec{\nu} \in \bar{G}\},\ ^{10}$$

where, as before, $\vec{\nu} \restriction \alpha := \langle \pi_{\mathrm{mc}(a^p_i), \mathrm{mc}(a^p_i \cap \alpha)}(\nu_i) \mid \ell(p) \leq i \leq |\vec{\nu}| \rangle$.

Claim 2.14.1. c_α *is well-defined.*

Proof. Let $c(\alpha, \vec{\nu} \restriction \alpha)_{\bar{G}_\alpha}$ and $d(\alpha, \vec{\eta} \restriction \alpha)_{\bar{G}_\alpha}$ be as above. These come with conditions $(p,c), (q,d) \in G$ with $\alpha \in \mathrm{dom}(c) \cap \mathrm{dom}(d)$.[11] Let $(r,e) \leq (p,c), (q,d)$ in G with $\ell(r) \geq \max\{\ell(p^\frown \vec{\nu}), \ell(q^\frown \vec{\eta})\}$. By definition, there are $\vec{\sigma}, \vec{\rho}$ such that $(r,e) \leq^* (p,c)^\frown \vec{\sigma}, (q,d)^\frown \vec{\rho}$. Clearly $\vec{\nu} \sqsubseteq \vec{\sigma}$ and $\vec{\eta} \sqsubseteq \vec{\rho}$.

Since $\alpha \in \mathrm{dom}(e)$ the definition of the $\leq^*$-order yields

$$(r \restriction \alpha)^\frown \vec{\pi} \Vdash_{\mathbb{P} \restriction \alpha} e(\alpha, \vec{\pi}) \leq (c^\frown \vec{\sigma})(\alpha, \vec{\pi} \restriction \alpha),$$

for all $\vec{\pi} \in \mathrm{dom}(e(\alpha))$. Here $\vec{\pi} \restriction \alpha$ is a shorthand for the sequence of projections of the π_i's via $\pi_{\mathrm{mc}(a^r_i \cap \alpha), \mathrm{mc}(a^p_i \cap \alpha)}$. The above is equivalent to

$$(r \restriction \alpha)^\frown \vec{\pi} \Vdash_{\mathbb{P} \restriction \alpha} e(\alpha, \vec{\pi}) \leq c(\alpha, \vec{\sigma} \restriction \alpha^\frown \vec{\pi} \restriction \alpha).$$

Notice that $(r \restriction \alpha)^\frown \vec{\pi}$ also forces $c(\alpha, \vec{\sigma} \restriction \alpha^\frown \vec{\pi} \restriction \alpha) \leq c(\alpha, \vec{\sigma} \restriction \alpha)$. Indeed,

$$(r \restriction \alpha)^\frown \vec{\pi} \leq (p \restriction \alpha)^\frown (\vec{\nu} \restriction \alpha^\frown \vec{\pi} \restriction \alpha)$$

and the latter forces the desired inequality (by definition of $\mathbb{M}$). Thereby, $(r \restriction \alpha)^\frown \vec{\pi}$ forces "$e(\alpha, \vec{\pi}) \leq c(\alpha, \vec{\nu} \restriction \alpha)$". By a similar argument, this condition also forces "$e(\alpha, \vec{\pi}) \leq d(\alpha, \vec{\eta} \restriction \alpha)$". Since this happens for an arbitrary $\vec{\pi}$ we may let one corresponding to the αth-projection of some $\vec{\tau}$ such that

[10] By density, there is always $(p,c) \in G$ with $\alpha \in \mathrm{dom}(c)$. Thus, c_α is not $\varnothing$.

[11] Note that, the notation $\vec{\nu} \restriction \alpha$ is imprecise in that it does not exhibit the dependence on p. We warn our readers that during this proof $\vec{\nu} \restriction \alpha$ (resp. $\vec{\eta} \restriction \alpha$) is obtained using the αth projection associated to p (resp. q).

$r^\frown\vec\tau \in \bar G$. In particular, $(r\restriction\alpha)^\frown\vec\pi \in \bar G_\alpha$ and so $c(\alpha,\vec\nu\restriction\alpha)_{\bar G_\alpha}$ and $d(\alpha,\vec\eta\restriction\alpha)_{\bar G_\alpha}$ are compatible conditions in $\mathrm{Add}(\kappa^+,1)_{V[\bar G_\alpha]}$. □

Let $\vec f := \langle f_\beta \mid \beta<\alpha\rangle$ be an injective enumeration of ${}^\kappa\kappa$ in $V[\bar G_\alpha]$. Define $\Phi\colon \alpha\to\kappa^+$ as $\Phi(\beta):=\gamma$ where γ is the least ordinal $<\kappa^+$ for which

$$c_\alpha(\gamma+\xi)=f_\beta(\xi), \text{ for all } \xi<\kappa.$$

The above definition is run inside $V[G]$. If well-defined, Φ establishes an injection between α and κ^+. In particular, α is collapsed to κ^+.[12]

Let us thus show that Φ is well-defined. For each $\beta<\alpha$ consider

$$D_\beta := \{(p,c)\in\mathbb M/\bar G \mid \exists\gamma<\kappa^+\,\forall\vec\pi,\xi\,(c(\alpha,\vec\pi)_{\bar G_\alpha}(\gamma+\xi)=f_\beta(\xi))\}.$$

Claim 2.14.2. *D_β is dense in $\mathbb M/\bar G$.*

Proof of claim. Let $(p,c)\in\mathbb M/\bar G$. Without loss of generality, $\alpha\in\mathrm{dom}(c)$. The definition of the forcing (see Definition 2.4(2)) implies that $c(\alpha,\vec\pi)_{\bar G_\alpha}\in \mathrm{Add}(\kappa^+,1)_{V[\bar G_\alpha]}$ for all $\vec\pi\in\mathrm{dom}(c(\alpha))$. Working in $V[\bar G_\alpha]$ let $\gamma<\kappa^+$ be above $\sup_{\vec\pi\in\mathrm{dom}(c(\alpha))}\mathrm{dom}(c(\alpha,\vec\pi)_{\bar G_\alpha})$. This choice is possible because there are at most κ-many possible $\vec\pi$ and κ^+ is a regular cardinal in $V[\bar G_\alpha]$. Let $c^* := \{\langle\gamma+\xi, f_\beta(\xi)\rangle \mid \xi<\kappa\}$ and note that $c^*\cup c(\alpha,\vec\pi)_{\bar G_\alpha}$ is a well-defined condition in $\mathrm{Add}(\kappa^+,1)_{V[\bar G_\alpha]}$. Moreover, adding c^* to the $c(\alpha,\vec\pi)_{\bar G_\alpha}$ does not conflict with the requirement that the conditions increase with larger $\vec\pi$ (Definition 2.4(3)). Let $\dot c^*$ be a $\mathbb P\restriction\alpha$-name such that $(\dot c^*)_{\bar G_\alpha}=c^*$. By modifying c^* is necessary we may assume that

$$\mathbb 1\Vdash_{\mathbb P\restriction\alpha}\dot c^*\cup c(\alpha,\vec\pi)\in \dot{\mathrm{Add}}(\kappa^+,1).$$[13]

Let d be the function with $\mathrm{dom}(d)=\mathrm{dom}(c)$, $\mathrm{dom}(d(\alpha))=\mathrm{dom}(c(\alpha))$ and for each $\vec\pi\in\mathrm{dom}(c(\alpha))$, $d(\alpha,\vec\pi):=\dot c^*\cup c(\alpha,\vec\pi)$. It is clear that $(p,d)\in\mathbb M/\bar G$, $(p,d)\le^*(p,c)$ and that it belongs to D_β. This shows that D_β is dense. □

Since G is also $\mathbb M/\bar G$-generic over $V[\bar G]$ we can let $(p,c)\in D_\beta\cap G$. It follows that for some γ, $c_\alpha(\gamma+\xi)=f_\beta(\xi)$ for all $\xi<\kappa$. So $\Phi(\beta)$ is defined. □

2.4. **Chain condition.** In this section we show that $\mathbb M$ does not collapse λ. This will be a consequence of the chain condition of $\mathbb M$. Since we are aiming for $\mathbb M$ to be the first step in an iteration *á-la-Σ-Prikry* we need to identify a stronger property which is iterable:

Definition 2.15. We say that $\mathbb M$ is *λ-Knaster to $<\lambda$-Linked* if for every $\mathcal X\in[\mathbb M]^\lambda$ there are $\mathcal Y\in[\mathcal X]^\lambda$, a regular cardinal $\delta<\lambda$ and a map

$$\mathfrak C\colon \{(p,c)^\frown\vec\nu \mid (p,c)\in\mathcal Y,\ \vec\nu\in\textstyle\prod_{\ell(p)\le n\le k}A^p_n,\ k\ge\ell(p)\}\to H_\delta$$

such that the following hold:

(†) $\mathfrak C((p,c)^\frown\vec\nu)=\mathfrak C((q,d)^\frown\eta)\ \Rightarrow\ \exists(r,d)\in\mathbb M\,((r,d)\le^*(p,c)^\frown\vec\nu,(q,d)^\frown\vec\eta)$.

[12]Recall that κ^+ is preserved by $\mathbb M$.

[13]For instance, by taking τ a $\mathbb P\restriction\alpha$-name forced by $p\restriction\alpha$ to be $\dot c^*$ and forced to be $\varnothing$ by conditions that are incompatible with $p\restriction\alpha$.

The idea behind the previous concept is the following. Given a collection of λ-many conditions $\mathcal{X}$ in $\mathbb{M}$ one can shrink it to a set $\mathcal{Y}$ which locally admits a compatibility function $\mathfrak{C}$. The fact that $\mathfrak{C}$ is defined over the minimal extensions of $(p,c)\in\mathcal{Y}$ and not just on $\mathcal{Y}$ has to do with the iterability of this property. For more details see the proof of Lemma 4.15. The rest of this section will be devoted to show that, in effect, $\mathbb{M}$ is λ-Knaster to $<\lambda$-Linked.

Lemma 2.16. *If $\mathbb{M}$ is λ-Knaster to $<\lambda$-Linked then it is λ-Knaster.*

In particular, if $\mathbb{M}$ is λ-Knaster to $<\lambda$-Linked then it forces "$\lambda=\kappa^{++}$".

Proof. Let $\mathcal{X}\in[\mathbb{M}]^{\lambda}$ and use that $\mathbb{M}$ is λ-Knaster to $<\lambda$-Linked to find $\mathcal{Y}\in[\mathcal{X}]^{\lambda}$ and a compatibility function $\mathfrak{C}$ as above. Since $\mathrm{Im}(\mathfrak{C})\subseteq H_\delta$ and λ is inaccessible we can find $\mathcal{Z}\subseteq\mathcal{Y}$ of size λ where $\mathfrak{C}$ is constant. □

Let $\delta\in(\kappa,\lambda)$ be an inaccessible cardinal. Arguing as in [Pov20, Definition 10.2.36] one can define a map $\mathfrak{c}_\delta\colon\mathbb{P}\restriction\delta\to H_\delta$ such that

$$\mathfrak{c}_\delta(p)=\mathfrak{c}_\delta(p')\Rightarrow\exists r\in\mathbb{P}\restriction\delta\;(r\leq^* p,q).$$

Let us now define a lifting of this map to the bigger poset $\mathbb{M}\restriction\delta$:

Definition 2.17. Let $\delta\in(\kappa,\lambda)$ be inaccessible. For $(p,c)\in\mathbb{M}\restriction\delta$ define

$$\mathfrak{c}_\delta(p,c):=\langle\mathfrak{c}_\delta(p^\frown\vec{\nu}),\alpha,c(\alpha,\vec{\nu}\restriction\alpha)\mid\alpha\in\mathrm{dom}(c),\,\vec{\nu}\restriction\alpha\in\mathrm{dom}(c(\alpha))\rangle,$$

where $\vec{\nu}\restriction\alpha:=\langle\pi_{\mathrm{mc}(a^p_i),\mathrm{mc}(a^p_i\cap\alpha)}(\nu_i)\mid\ell\leq i\leq k\rangle$.

Remark 2.18. $\mathfrak{C}_\delta$ has range in H_δ. Indeed $\mathrm{dom}(c)\in[\delta]^{\leq\kappa}$ and every $\mathbb{P}\restriction\alpha$-name $c(\alpha,\vec{\nu}\restriction\alpha)$ can be coded inside H_δ because $\mathbb{P}\restriction\alpha\in H_\delta$.

Lemma 2.19. *$\mathbb{M}$ is λ-Knaster to $<\lambda$-Linked.*

Proof. Let $\{(p_\xi,c_\xi)\mid\xi<\lambda\}\subseteq\mathbb{M}$. Following Gitik [Git10, Lemma 2.15] one may assume that for each $\xi<\lambda$ the following hold:

- $\ell(p_\xi)=\ell$;
- $f^{p_\xi}_n\cup f^{p_\zeta}_n$ is a function for all $n<\omega$;
- for each $n\geq\ell$, $\{\mathrm{dom}(f^{p_\xi}_n)\cup a^{p_\xi}_n\mid\xi<\lambda\}$ forms a Δ-system with root Δ_n satisfying that $\mathrm{dom}(f^{p_\xi}_n)\cap a^{p_\zeta}_n=\emptyset$.

In addition, we may assume that

(1) for each $n\geq\ell$ and $\xi<\lambda$, $A^{p_\xi}_n=A_n$,
(2) for each $n\geq\ell$, $\{\pi_{\mathrm{mc}(a^{p_\xi}_n),*}\restriction(\Delta_n\cap a^{p_\xi}_n)\times A_n\mid\xi<\lambda\}$ is a singleton.[14]

Claim 2.19.1. *For each $\xi,\zeta<\lambda$, p_ξ and p_ζ are $\leq^*$-compatible.*

Moreover, for each $\vec{\nu}\in[\prod_{\ell\leq n}A_n]^{<\omega}$, $p_\xi^\frown\vec{\nu}$ and $p_\zeta^\frown\vec{\nu}$ are $\leq^*$-compatible.

Proof. The assertion that p_ξ and p_ζ are $\leq^*$-compatible is immediate. For the moreover claim note that the incompatibilities between $p_\xi^\frown\vec{\nu}$ and $p_\zeta^\frown\vec{\nu}$ must arise in the form of discrepancies between $\pi_{\mathrm{mc}(a^{p_\xi}_n),\delta}(\nu_n)$ and $\pi_{\mathrm{mc}(a^{p_\zeta}_n),\delta}(\nu_n)$ for δ's in Δ_n. However, these two values are the same by (2) above. □

[14]Here $\pi_{\mathrm{mc}(a^{p_\xi}_n),*}$ denotes the map defined as $(\delta,\nu)\mapsto\pi_{\mathrm{mc}(a^{p_\xi}_n),\delta}(\nu)$.

For each $\xi < \lambda$, $\mathrm{dom}(c_\xi)$ is a subset of λ of cardinality $\leq\kappa$. So, by further shrinking the indexing set, we may assume that $\{\mathrm{dom}(c_\xi) \mid \xi < \lambda\}$ is a Δ-system; say, with root Δ. Let $\delta \in (\kappa, \lambda)$ be an inaccessible cardinal with $\Delta \subseteq \delta$. By the forthcoming Lemma 2.20, $\{(p_\xi \restriction \delta, c_\xi \restriction \delta) \mid \alpha < \lambda\}$ are conditions in $\mathbb{M} \restriction \delta$. For each $\alpha < \lambda$ define

$$\mathfrak{C}((p_\xi, c_\xi)^\frown \vec{\nu})) := \langle \vec{\nu}, \mathfrak{c}_\delta(p_\xi \restriction \delta, c_\xi \restriction \delta)\rangle,$$

where $\mathfrak{c}_\delta$ is as in Definition 2.17. Clearly, $\mathfrak{C}$ has range in H_δ.

We claim that $\mathfrak{C}$ satisfies equation (†) of Definition 2.15. Suppose that $\mathfrak{C}((p_\xi, c_\xi)^\frown \vec{\nu})) = \mathfrak{C}((p_\zeta, c_\zeta)^\frown \vec{\eta}))$. Then $\vec{\nu} = \vec{\eta}$ and by Claim 2.19.1 both $p_\xi{}^\frown \vec{\nu}$ and $p_\zeta{}^\frown \vec{\nu}$ are $\leq^*$-compatible. Let r be $\leq^*$-stronger and d be the function with domain $\mathrm{dom}(c_\xi) \cup \mathrm{dom}(c_\zeta)$ such that

$$\forall \vec{\pi} \in \mathrm{dom}(d(\alpha)) = [\textstyle\prod_{n \geq \ell(r)} \pi_{\mathrm{mc}(a^r_n), \mathrm{mc}(a^r_n \cap \alpha)} “A^r_n]^{<\omega},$$

for each $\alpha \in \mathrm{dom}(d) \setminus \Delta$ (outside the common domain),

$$d(\alpha, \vec{\pi}) := \begin{cases} (c_\xi{}^\frown \vec{\nu})(\alpha, \vec{\pi}_{\xi,\alpha}), & \text{if } \alpha \in \mathrm{dom}(c_\xi); \\ (c_\zeta{}^\frown \vec{\nu})(\alpha, \vec{\pi}_{\zeta,\alpha}) & \text{if } \alpha \in \mathrm{dom}(c_\zeta). \end{cases}$$

If $\alpha \in \Delta$, $d(\alpha, \vec{\pi}) := (c_\xi{}^\frown \vec{\nu})(\alpha, \vec{\pi}_{\xi,\alpha})$. Above we used the following notation:

$$\vec{\pi}_{x,\alpha} := \langle \pi_{\mathrm{mc}(a^r_i \cap \alpha), \mathrm{mc}(a^{p_x}_i \cap \alpha)}(\pi_i) \mid \ell(r) \leq i \leq k\rangle \text{ for } x \in \{\xi, \zeta\}.$$

Clearly (r, d) is a condition in $\mathbb{M}$ and $(r, d) \leq^* (p_\xi, c_\xi)^\frown \vec{\nu}$.

The reason (r, d) why is $\leq^*$-stronger than $(p_\zeta, c_\zeta)^\frown \vec{\nu}$ is that $(c_\xi{}^\frown \vec{\nu})(\alpha, \vec{\pi}_{\xi,\alpha}) = (c_\zeta{}^\frown \vec{\nu})(\alpha, \vec{\pi}_{\zeta,\alpha})$ whenever $\alpha \in \Delta$. To see this, fix $\vec{\pi} \in \mathrm{dom}(d(\alpha))$ and let $\vec{\sigma}$ be a preimage of it in the measure one sets of r. Then,

$$(r, d)^\frown \vec{\sigma} \leq^* (p_\xi, c_\xi)^\frown \vec{\nu}^\frown \vec{\sigma}_\xi, (p_\zeta, c_\zeta)^\frown \vec{\nu}^\frown \vec{\sigma}_\zeta,$$

where $\vec{\sigma}_\xi$ (resp. $\vec{\sigma}_\zeta$) is a shorthand for $\vec{\sigma}_{\xi,\lambda}$ (resp. $\vec{\sigma}_{\zeta,\lambda}$). In particular, the same holds when projecting down to δ. Let $\vec{\nu}(\xi), \vec{\sigma}(\xi)$ be the sequence defined by the projections

$$\pi_{\mathrm{mc}(a^{p_\xi}_i), \mathrm{mc}(a^{p_\xi}_i \cap \delta)}(\nu_i) \text{ and } \pi_{\mathrm{mc}(a^{p_\xi}_i), \mathrm{mc}(a^{p_\xi}_i \cap \delta)}(\sigma_\xi(i)).$$

Define $\vec{\nu}(\zeta)$ and $\vec{\sigma}(\zeta)$ similarly using the projections associated to p_ζ.

Since $\mathfrak{c}_\delta(p_\xi \restriction \delta, c_\xi \restriction \delta) = \mathfrak{c}_\delta(p_\zeta \restriction \delta, c_\zeta \restriction \delta)$ it follows, by Definition 2.17, that $\mathfrak{c}_\delta((p_\xi \restriction \delta)^\frown \langle \vec{\nu}(\xi)^\frown \vec{\sigma}(\xi)\rangle) = \mathfrak{c}_\delta(q)$ for the unique condition q that is: 1) a minimal extension of $p_\zeta \restriction \delta$; 2) compatible with $(p_\xi \restriction \delta)^\frown \langle \vec{\nu}(\xi)^\frown \vec{\sigma}(\xi)\rangle$; and 3), has the same length as $(p_\xi \restriction \delta)^\frown \langle \vec{\nu}(\xi)^\frown \vec{\sigma}(\xi)\rangle$. It must be the case that

$$q = (p_\zeta \restriction \delta)^\frown \langle \vec{\nu}(\zeta)^\frown \vec{\sigma}(\zeta)\rangle,$$

Using again Definition 2.17 and the commutativity of the projections one can check that $(c_\xi{}^\frown \vec{\nu})(\alpha, \vec{\pi}_{\xi,\alpha}) = (c_\zeta{}^\frown \vec{\nu})(\alpha, \vec{\pi}_{\zeta,\alpha})$. □

2.5. **Projections.** In this section we show that there is a natural projection between $\mathbb{M}$ and $\mathbb{M}\restriction\alpha$ for all regular cardinals α in $[\kappa^+,\lambda)$ (i.e., $\alpha\in\mathcal{R}$). This will be instrumental in our later proof of the ineffable tree property at λ.

Lemma 2.20. *For each $\alpha\in\mathcal{R}$ the map $(p,c)\mapsto(p\restriction\alpha,c\restriction\alpha)$ defines a length-preserving projection from $\mathbb{M}$ to $\mathbb{M}\restriction\alpha$.*

Proof. First, note that $c\restriction\alpha$ has the right shape and thus $(p\restriction\alpha,c\restriction\alpha)\in\mathbb{M}\restriction\alpha$. Indeed, this follows from commutativity of the projections; namely,

$$\pi_{\mathrm{mc}(a^p_i),\mathrm{mc}(a^p_i\cap\beta)}``A^p_i=\pi_{\mathrm{mc}(a^p_i\cap\alpha),\mathrm{mc}(a^p_i\cap\beta)}``A^{p\restriction\alpha}_i.^{15}$$

Suppose $(p,c)\leq(q,d)$ and let $\vec{\nu}$ be with $(p,c)\leq^*(q,d)^\frown\vec{\nu}$. We show that

$$(p\restriction\alpha,c\restriction\alpha)\leq^*(q\restriction\alpha,d\restriction\alpha)^\frown\langle\pi_{\mathrm{mc}(a^q_i),\mathrm{mc}(a^q_i\cap\alpha)}(\nu_i)\mid i\leq|\vec{\nu}|\rangle.$$

For this it suffices to check that

$$(p\restriction\beta)^\frown\vec{\pi}\Vdash_{\mathbb{P}\restriction\beta}c(\beta,\vec{\pi})\leq(d^\frown\vec{\nu})(\beta,\langle\pi_{\mathrm{mc}(a^p_i\cap\beta),\mathrm{mc}(a^q_i\cap\beta)}(\pi_i)\mid\ell(p)\leq i\leq k\rangle)$$

for all $\beta\in\mathrm{dom}(d)\cap\alpha$ and $\vec{\pi}\in\prod_{\ell(p)\leq i\leq k}\pi_{\mathrm{mc}(a^p_i\cap\alpha),\mathrm{mc}(a^p_i\cap\beta)}``A^{p\restriction\alpha}_i$. However, this happens automatically. Indeed, it follows from $(p,c)\leq^*(q^\frown\vec{\nu},d^\frown\vec{\nu})$ (see Clause (III) of Definition 2.4).

Let us finally check that the map is a projection. Let $(q,d)\leq(p\restriction\alpha,c\restriction\alpha)$. First, let $r\leq p$ be such that $r\restriction\alpha\leq^* q$. Say $\vec{\nu}$ witnesses that $r\leq^* p^\frown\vec{\nu}$. Define the *Mitchell-part* of r as follows. Let e be the function with domain $\mathrm{dom}(e):=\mathrm{dom}(d)\cup(\mathrm{dom}(c)\setminus\alpha)$ such that for each $\beta\in\mathrm{dom}(e)$:

- $\mathrm{dom}(e(\beta)):=[\prod_{i\geq\ell(r)}\pi_{\mathrm{mc}(a^r_i),\mathrm{mc}(a^r_i\cap\beta)}``A^r_i]^{<\omega}$;
- for each $\vec{\pi}=\langle\pi_{\ell(r)},\dots,\pi_k\rangle\in\mathrm{dom}(e(\beta))$,

$$e(\beta,\vec{\pi}):=\begin{cases}(c^\frown\vec{\nu})(\beta,\langle\pi_{\mathrm{mc}(a^r_i\cap\beta),\mathrm{mc}(a^p_i\cap\beta)}(\pi_i)\mid\ell(r)\leq i\leq k\rangle), & \text{if }\beta\notin\mathrm{dom}(d);\\ d(\beta,\langle\pi_{\mathrm{mc}(a^r_i\cap\beta),\mathrm{mc}(a^q_i\cap\beta)}(\pi_i)\mid\ell(r)\leq i\leq k\rangle), & \text{if }\beta\in\mathrm{dom}(d).\end{cases}$$

It is clear that $\mathbb{1}_{\mathbb{P}\restriction\beta}$ forces $e(\beta,\vec{\pi})$ to be a condition in $\mathrm{Add}(\kappa^+,1)$ and that $e(\beta,\vec{\pi})$ becomes stronger with longer $\vec{\pi}$'s. Namely, $(r,e)\in\mathbb{M}$.

Let us check that $(r,e)\leq(p,c)$ and that $(r\restriction\alpha,e\restriction\alpha)\leq^*(q,d)$. Looking at Definition 2.4(III), the latter is pretty obvious. The former also holds in that $(r,e)\leq^*(p,c)^\frown\vec{\nu}$. To see this, note that for $\beta\in\mathrm{dom}(c)\setminus\alpha$ the verification is obvious. For $\beta\in\mathrm{dom}(d)$ and $\vec{\pi}=\langle\pi_{\ell(r)},\dots,\pi_k\rangle$ in $\prod_{\ell(r)\leq i\leq k}\pi_{\mathrm{mc}(a^r_i),\mathrm{mc}(a^r_i\cap\beta)}``A^r_i$ we need to check that

$$(r\restriction\beta)^\frown\vec{\pi}\Vdash_{\mathbb{P}\restriction\beta}e(\beta,\vec{\pi})\leq(c^\frown\vec{\nu})(\beta,\langle\pi_{\mathrm{mc}(a^r_i\cap\beta),\mathrm{mc}(a^p_i\cap\beta)}(\pi_i)\mid\ell(r)\leq i\leq k\rangle).$$

To argue this we will use that $(q,d)\leq(p\restriction\alpha,c\restriction\alpha)$. Let $\vec{\sigma}$ be such that $(q,d)\leq^*(p\restriction\alpha,c\restriction\alpha)^\frown\vec{\sigma}$. Necessarily $\vec{\sigma}=\langle\pi_{\mathrm{mc}(a^p_i),\mathrm{mc}(a^p_i\cap\alpha)}(\nu_i)\mid i<|\vec{\nu}|\rangle$. By

[15] $A^{p\restriction\alpha}_i$ is by definition $\pi_{\mathrm{mc}(a^p_i),\mathrm{mc}(a^p_i\cap\alpha)}``A^p_i$.

definition we have that $(q \restriction \beta)^\frown \langle \pi_{\mathrm{mc}(a_i^r\cap\beta),\mathrm{mc}(a_i^q\cap\beta)}(\pi_i) \mid \ell(r) \le i \le k\rangle$ forces

$$d(\beta, \langle \pi_{\mathrm{mc}(a_i^r\cap\beta),\mathrm{mc}(a_i^q\cap\beta)}(\pi_i) \mid \ell(r) \le i \le k\rangle) \le$$
$$((c \restriction \alpha)^\frown \vec{\sigma})(\beta, \langle \pi_{\mathrm{mc}(a_i^r\cap\beta),\mathrm{mc}(a_i^p\cap\beta)}(\pi) \mid \ell(r) \le i \le k\rangle).$$

Since $\vec{\sigma}$ and $\vec{\nu}$ project the same way below α a moment of reflection makes clear that $((c \restriction \alpha)^\frown \sigma)(\beta, \cdot) = (c^\frown \vec{\nu})(\beta, \cdot)$. Thus, the above condition forces

$$d(\beta, \langle \pi_{\mathrm{mc}(a_i^r\cap\beta),\mathrm{mc}(a_i^q\cap\beta)}(\pi_i) \mid \ell(r) \le i \le k\rangle) \le$$
$$(c^\frown \vec{\nu})(\beta, \langle \pi_{\mathrm{mc}(a_i^r\cap\beta),\mathrm{mc}(a_i^p\cap\beta)}(\pi) \mid \ell(r) \le i \le k\rangle).$$

Since $(r \restriction \beta)^\frown \vec{\pi} \le (q \restriction \beta)^\frown \langle \pi_{\mathrm{mc}(a_i^r\cap\beta),\mathrm{mc}(a_i^q\cap\beta)}(\pi_i) \mid \ell(r) \le i \le k\rangle$ it follows that $(r \restriction \beta)^\frown \vec{\pi} \le (q \restriction \beta)$ forces the desired property. □

Let us close this section analyzing the subforcing of $\mathbb{M}$ where the Prikry part is fixed. Formally, let π be the natural projection between $\mathbb{M}$ and $\mathbb{P}$; namely, $\pi\colon (p, c) \mapsto p$. For each $n < \omega$ let us denote by $\mathbb{M}_n^\pi$ the subforcing of $\mathbb{M}_n := \{(p, c) \in \mathbb{M} \mid \ell(p)\}$ endowed with the following order:

$$(p, c) \le^\pi (q, d) \text{ iff } (p, c) \le^* (q, d) \text{ and } p = q.$$

Lemma 2.21. *For each $n < \omega$, $\mathbb{M}_n^\pi$ is κ^+-directed-closed.*

Proof. Let $D \subseteq \mathbb{M}_n^\pi$ be a $\le^\pi$-directed set of conditions of size $\le\kappa$. Let p be the common EBPF-part of conditions in D. Let c be the function with

- $\mathrm{dom}(c) := \bigcup_{(p,d)\in D} \mathrm{dom}(d)$,
- $\mathrm{dom}(c(\alpha)) := [\prod_{n\ge\ell(p)} \pi_{\mathrm{mc}(a_n^p),\mathrm{mc}(a_n^p\cap\alpha}``A_n^p]^{<\omega}$,
- $c(\alpha, \vec{\pi}) := \bigwedge\{d(\alpha, \vec{\pi}) \mid \alpha \in \mathrm{dom}(d)\}$,

where this lower bound is forced by $(p\restriction\alpha)^\frown\vec{\pi}$ to be such. As usual, we tweak $c(\alpha, \vec{\pi})$ a bit so that $\mathbb{1}_{\mathbb{P}\restriction\alpha}$ forces it to be a condition $\dot{\mathrm{A}}\mathrm{dd}(\kappa^+, 1)$. Clearly, (p, c) is a $\le^\pi$-lower bound for D. □

2.6. **Conclusion.** With all the previous results at hand it is easy to verify that $\mathbb{M}$ satisfies all the axioms of the Σ-Prikry framework (see [PRS22, §2]) with the only exception of the existence of a compatibility function c. Remember that in the current scenario the preservation of cardinals $\ge\lambda$ is handled by Lemma 2.19. Let us call this family *weak-Σ-Prikry*.

The following is a summary of the main properties of $\mathbb{M}$:

Corollary 2.22. *The following properties are true for $\mathbb{M}$:*

(1) $\mathbb{M}$ is weak-Σ-Prikry having property $\mathcal{D}$;
(2) $\mathbb{M}$ forces $2^\kappa = \lambda$ and preserves cardinals except those in (κ^+, λ).
(3) For each V-regular cardinal $\alpha \in [\kappa^+, \lambda)$ the map $(p, c) \mapsto (p\restriction\alpha, c\restriction\alpha)$ defines a length-preserving projection from $\mathbb{M}$ to $\mathbb{M}\restriction\alpha$.

3. The Mitchell-like Extender Based forcing with collapses

In this section we introduce a variation of the poset of Definition 2.4 which brings down to $\aleph_\omega$ the cardinal structure of the final generic extension. We follow very much the structure of the previous section: In §3.1 we give the definition of Gitik's forcing from [Git19] and discuss some of its aspects that will be relevant later on. In §3.2 we present our forcing. In §3.3 we show that it has the Prikry property and in §3.4 we analyze its associated cardinal structure. §3.5 discusses the projections of the poset. §3.6 closes the section by summarizing the main properties of the forcing.

3.1. **Gitik's EBPF with collapses.** In this section we consider a version of our Mitchell EBPF which brings down to $\aleph_\omega$ the cardinal structure of the corresponding generic extension. For this we replace the Gitik-Magidor EBPF by *Gitik's* EBPF *with collapses* [Git19] in $\mathbb{M}$. Our presentation follows [PRS21, §4] modulo some mild adjustments (see Remark 3.8).

We maintain the blanket assumptions of §2 (see p.3) and set $\kappa_{-1} := \aleph_1$. Before giving the definition of Gitik's forcing we first introduce its basic building block modules $\mathbb{Q}_{n0}$ and $\mathbb{Q}_{n1}$. Let $s_n\colon \kappa_n \to \kappa_n$ representing κ^+ using the *normal generator*, κ_n. Specifically, for each $n < \omega$, $j_{E_n}(s_n)(\kappa_n) = \kappa^+$.

Definition 3.1. For each $n < \omega$, define $\mathbb{Q}_{n1}$, $\mathbb{Q}_{n0}$ and $\mathbb{Q}_n$ as follows:

$(0)_n$ $\mathbb{Q}_{n0} := (Q_{n0}, \leq_{n0})$ is the set of $p := (a^p, A^p, f^p, F^{0p}, F^{1p}, F^{2p})$, where:
  (1) (a^p, A^p, f^p) is in the $n0$-module Q^*_{n0} from the EBPF as defined in [Git10, Definition 2.6]. Moreover, we require that $\kappa_n, \kappa^+ \in a^p$ and that $a^p \cap \alpha$ contains a $\leq_{E_n}$-greatest element for all $\alpha \in \mathcal{R}$.[16]
  (2) For $i < 3$, $\mathrm{dom}(F^{ip}) = A^p$ and for $\nu \in \mathrm{dom}(F^{ip})$,
    (a) $F^{0p}(\nu) \in \mathrm{Col}(\kappa^{+4}_{n-1}, {<}\nu_0)$;
    (b) $F^{1p}(\nu) \in \mathrm{Col}(\nu, s_n(\nu_0))$;
    (c) $F^{2p}(\nu) \in \mathrm{Col}(s_n(\nu_0)^{++}, {<}\kappa_n)$,

  where $\nu_0 := \pi_{\mathrm{mc}(a^p),\kappa_n}(\nu)$. The ordering $\leq_{n0}$ is defined as follows: $q \leq_{n0} p$ iff $(a^q, A^q, f^q) \leq_{\mathbb{Q}^*_{n0}} (a^p, A^p, f^p)$ as in [Git10, Definition 2.7], and for each $\nu \in \mathrm{dom}(F^{iq})$, $F^{iq}(\nu) \supseteq F^{ip}(\nu')$, where $\nu' := \pi_{\mathrm{mc}(a^q),\mathrm{mc}(a^p)}(\nu)$.

$(1)_n$ $\mathbb{Q}_{n1} := (Q_{n1}, \leq_{n1})$ is the set of $p := (f^p, \rho^p, h^{0p}, h^{1p}, h^{2p})$, where:
  (1) f^p is a function from some $x \in [\lambda]^{\leq\kappa}$ to κ_n;
  (2) $\rho^p < \kappa_n$ inaccessible;
  (3) $h^{0p} \in \mathrm{Col}(\kappa^{+4}_{n-1}, {<}\rho^p)$;
  (4) $h^{1p} \in \mathrm{Col}(\rho^p, s_n(\rho^p))$;
  (5) $h^{2p} \in \mathrm{Col}(s_n(\rho^p)^{++}, {<}\kappa_n)$.

  The ordering $\leq_{n1}$ is defined as follows: $q \leq_{n1} p$ iff $f^q \supseteq f^p$, $\rho^p = \rho^q$, and for $i < 3$, $h^{iq} \supseteq h^{ip}$.

$(2)_n$ Set $\mathbb{Q}_n := (Q_{n0} \cup Q_{n1}, \leq_n)$ where the ordering $\leq_n$ is defined as follows: for each $p, q \in Q_n$, $q \leq_n p$ iff

[16] Recall Lemma 2.2.

(1) either $p, q \in Q_{ni}$, some $i \in \{0,1\}$, and $q \leq_{ni} p$, or
(2) $q \in Q_{n1}$, $p \in Q_{n0}$ and, for some $\nu \in A^p$, $q \leq_{n1} p^\frown\langle\nu\rangle$, where

$$p^\frown\nu := (f^p \cup \{\langle\beta, \pi_{\mathrm{mc}(a^p),\beta}(\nu)\rangle \mid \beta \in a^p\}, \nu, F^{0p}(\nu), F^{1p}(\nu), F^{2p}(\nu)).$$

Having all necessary building blocks we can now define Gitik's poset $\mathbb{P}$:

Definition 3.2. The Extender Based Prikry Forcing with collapses (EBPFC) is the poset $\mathbb{P}$ consisting of sequences $p = \langle p_n \mid n < \omega\rangle \in \prod_{n<\omega} Q_n$ such that

(1) There is $\ell(p) < \omega$ such that $p_n \in Q_{n0}$ for all $n \geq \ell(p)$;
(2) For every $\ell(p) \leq n \leq m$, $a^{p_n} \subseteq a^{p_m}$.

Given $p, q \in \mathbb{P}$ write $p \leq^* q$ if $\ell(p) = \ell(q)$ and $p_n \leq_n q_n$ for all $n < \omega$.

One-point extensions of a condition are defined in the expected manner:

Definition 3.3. Given $p \in \mathbb{P}$ and $\nu \in A^p_\ell$ define $p^\frown\nu$ as

$$\langle p_0, \dots, p_{\ell(p)-1}, {p_{\ell(p)}}^\frown\nu, p_{\ell(p)+1}, \dots\rangle,$$

where $p_{\ell(p)}{}^\frown\nu$ is as in Clause $(2)_n$ above. In general, for $\vec{\nu} \in \prod_{\ell(p)\leq i\leq n} A^p_i$ one defines $p^\frown\vec{\nu}$ by recursion in the obvious fashion.

Note that $p^\frown\vec{\nu}$ is the $\leq^*$-greatest extension of p with length $\ell(p) + |\vec{\nu}|$ where the new *Prikry points* are taken from $\vec{\nu}$.

The regular order of $\mathbb{P}$ is defined as usual; namely, as a combination of the *pure extension ordering* $\leq^*$ and the ${}^\frown\vec{\nu}$-operation:

Definition 3.4 (The forcing ordering)**.** Given $p, q \in \mathbb{P}$ write $p \leq q$ if there is $\vec{\nu} \in \prod_{\ell(q)\leq i\leq n} A^q_i$ such that $p \leq^* q^\frown\vec{\nu}$.

There is a natural strengthening of the $\leq^*$-ordering enjoying of better closure properties – this is the so-called *the fusion ordering*.

Definition 3.5 (Fusion ordering)**.** Let $p, q \in \mathbb{P}$ be conditions with length ℓ and $k \geq \ell$. We shall write $p \leq^{*,k} q$ if $p \leq^* q$ and the following clauses hold:

(1) $\vec{h}^p_n = \vec{h}^q_n$ and $\rho^p_n = \rho^q_n$ for $n < \ell$;
(2) $a^p_n = a^q_n$ for $n \in [\ell, k]$;
(3) $A^p_n = A^q_n$ for $n \in [\ell, k]$;
(4) $\vec{F}^p_n = \vec{F}^q_n$ for $n \in [\ell, k]$.

We shall write $p \leq^{*,k,-} q$ if $p \leq^{*,k} q$ hold with the possible exception of Clause (3) above (i.e., the measure one sets A^p_n and A^q_n might not agree).

Lemma 3.6. *For each $k < \omega$, $\langle\mathbb{P}, \leq^{*,k}\rangle$ is κ_k^{+4}-closed.* □

Remark 3.7. For the usual Extender-Based Prikry forcing $\mathbb{P}$ (without collapses) $p \leq^{*,k} q$ (resp. $p \leq^{*,k,-} q$) amounts to saying that $p \leq^* q$ and that both have the same (a, A)-parts (resp. a-part) up to coordinate k. Note that in this case the $\leq^{*,k}$-ordering enjoys of a better closure property; namely, $\langle\mathbb{P}, \leq^{*,k}\rangle$ is κ_{k+1}-closed. The orderings $\leq^{*,k,-}$ and $\leq^{*,k}$ will play a key role in our proof of the tree property in §6 and §7.

Remark 3.8. Readers familiar with Gitik's forcing may have noted two differences with respect to the definition given in [Git19]. First, the domains of the F^{ip}'s are A^p in place of $\pi_{\mathrm{mc}(a^p),\mathrm{mc}(a^p\cap\kappa^+)}$ "A^p; second, the functions F^{0p} and h^{0p} start collapsing cardinals past κ_{n-1}^{+4} instead of κ_{n-1}^{+}. There are two reasons behind the adoption of these changes. Namely,

Reason 1: Our first modification is needed for the following natural strengthening of the *Strong Prikry Property* (SPP):

Lemma 3.9. *For each $p \in \mathbb{P}$, $D \subseteq \mathbb{P}$ dense open and $k \geq \ell(p)$ there is $q \leq^{*,k,-} p$ and $n < \omega$ such that every n-step extension of q is in D.*

This enhancement of the SPP holds for the usual Extender-Based Prikry forcing but seems not to be true for the version of Gitik's forcing given in [Git19]. This becomes evident inspecting our proof in [PRS21, Lemma 4.21].

Reason 2: The extra closure of the collapsing maps h_0^{pi}'s and F_0^{pi}'s will be used in the proof of the *Splitting Lemma* (Lemma 6.8), the main ingredient to establish the Tree Property at $\aleph_{\omega+2}$.

Given $p \in \mathbb{P}$, $n \geq \ell(p)$ and a regular cardinal $2^{\kappa_n} < \delta_n < \kappa_n^{+4}$:

▶ The Levy collapses $\vec{h} := \langle h_i^{p,j} \mid i < \ell(p)\rangle$ belong to a set of cardinality $<\delta_n$. Thus, we may eventually stabilize if we are given δ_n-many such objects.

▶ The promises $\langle F_i^{p,j} \mid i > n\rangle$ are at least κ_n^{+4}-closed. Since $\delta_n < \kappa_n^{+4}$ we may take lower bounds given δ_n-many such objects.

▶ Finally, the maps $\langle F_i^{pj} \mid i \in [\ell(p), n]\rangle$ can be also stabilized: These are partial functions $F_i^{pj} \colon \kappa_i \to V_{\kappa_i}$ with domain A_i^p so if we have $\langle p_\alpha \mid \alpha < \delta_n\rangle$ with $A_i^{p_\alpha} = A_i^{p_\beta}$ then one can find δ_n-many p_α's with the same F_i^{pj}'s. □

As in the previous section, there is a natural projection between $\mathbb{P}$ and $\mathbb{P} \restriction \alpha$ whenever $\alpha \in \mathcal{R}$. Specifically, for each $p \in \mathbb{P}$ and $\alpha \in \mathcal{R}$ define

$$p \restriction \alpha := \langle p_n \restriction \alpha \mid n < \omega\rangle,$$

where

$$p_n \restriction \alpha := \begin{cases} (f_n^p \restriction \alpha, \rho_n^p, h_n^{0p}, h_n^{1p}, h_n^{2p}), & \text{if } n < \ell(p); \\ (a_n^p \cap \alpha, A_n^p \restriction \alpha, f_n^p \restriction \alpha, F_n^{0p} \restriction \alpha, F_n^{1p} \restriction \alpha, F_n^{2p} \restriction \alpha), & \text{if } n \geq \ell(p), \end{cases}$$

where $A_n^p \restriction \alpha := \pi_{\mathrm{mc}(a_n^p),\mathrm{mc}(a_n^p\cap\alpha)}$ "A_n^p and $F_n^{ip} \restriction \alpha := F_n^{ip} \restriction (A_n^p \restriction \alpha)$.

Notation 3.10. We will use the following notation from [PRS21, §4]. For each $n < \omega$, $\mathbb{S}_n$ denotes the product of the first n-many collapses; more precisely, conditions in $\mathbb{S}_n$ are n-tuples $\langle(\rho_k^p, h_k^{0p}, h_k^{1p}, h_k^{2p}) \mid k < n\rangle$ for some $p \in \mathbb{P}$ with length $\geq n$; $\preceq$ stands for $\leq_{\mathbb{S}_n}$.

This induces maps $\varpi_n \colon \mathbb{P}_{\geq n} \to \mathbb{S}_n$ extracting the first n-many collapses of a condition of length $\geq n$. We have that ϖ_n is a *nice projection* ([PRS21, Definition 2.2]) as shown in [PRS21, Lemma 4.23]. So, given $p \in \mathbb{P}_{\geq n}$ and $t \preceq \varpi_n(p)$, there is a condition $p + t$ which is the $\leq^*$-greatest $q \leq^* p$ with

$\varpi_n(q) = t$. More explicitly, $p + t$ is the condition resulting from replacing the first n-many collapses of p by t.

Finally, $\mathbb{S} \downarrow s$ denotes all conditions in $\mathbb{S}$ stronger than s.

3.2. **Mitchell-like EBPF with collapses.** Let us now show how to interleave collapses within the Mitchell-like forcing $\mathbb{M}$ from Definition 2.4. To avoid cumbersome notations we keep denoting the modified poset by $\mathbb{M}$.

Definition 3.11. A condition in $\mathbb{M}$ is a pair (p, c) where $p \in \mathbb{P}$, c is a function with $\mathrm{dom}(c) \in [\mathcal{R}]^{\leq\kappa}$ and for each $\alpha \in \mathrm{dom}(c)$ the following hold:

(1) $c(\alpha)\colon \mathrm{dom}(c(\alpha)) \to V^{\mathbb{P}\restriction\alpha}$ where $\mathrm{dom}(c(\alpha))$ consists of pairs $\langle \vec{\pi}, t\rangle$, such that for some $k \geq \ell(p)$,

$$\vec{\pi} \in \prod_{\ell(p)\leq n\leq k} \pi_{\mathrm{mc}(a^p_i),\mathrm{mc}(a^p_i\cap\alpha)}\text{“}A^p_i \ \wedge\ t \in \bigcup_{n\leq k} \mathbb{S}_n \downarrow \varpi_n(p\restriction \alpha^\frown\vec{\pi}).$$

(2) $\mathbb{1} \Vdash_{\mathbb{P}\restriction\alpha}$ “$c(\alpha, \vec{\pi}, t) \in \dot{\mathrm{Add}}(\kappa^+, 1)$”, for all $\langle \vec{\pi}, t\rangle \in \mathrm{dom}(c(\alpha))$;

(3) For all $\langle \vec{\sigma}, s\rangle, \langle \vec{\pi}, t\rangle \in \mathrm{dom}(c(\alpha))$ with $(p \restriction \alpha)^\frown\vec{\sigma} + s \leq (p \restriction \alpha)^\frown\vec{\pi} + t$,

$$(p \restriction \alpha)^\frown\vec{\sigma} + s \Vdash_{\mathbb{P}\restriction\alpha} c(\alpha, \vec{\sigma}, s) \leq c(\alpha, \vec{\pi}, t).$$

Given $(p, c), (q, d) \in \mathbb{M}$ we write $(p, c) \leq^* (q, d)$ if and only if:

(I) $p \leq^* q$;

(II) $\mathrm{dom}(c) \supseteq \mathrm{dom}(d)$;

(III) for $\alpha \in \mathrm{dom}(d)$ and $\langle\langle \pi_\ell, \dots, \pi_k\rangle, t\rangle \in \mathrm{dom}(c(\alpha))$

$$(p\restriction\alpha)^\frown\vec{\pi}+t \Vdash_{\mathbb{P}\restriction\alpha} \text{“}c(\alpha, \vec{\pi}, t) \leq d(\alpha, \langle \pi_{\mathrm{mc}(a^p_i\cap\alpha),\mathrm{mc}(a^q_i\cap\alpha)}(\pi_i) \mid \ell \leq i \leq k\rangle, \mathbf{Col}(p \restriction \alpha, \vec{\pi}, t))\text{”},$$

where $\mathbf{Col}(p \restriction \alpha, \vec{\pi}, t)$ denotes the collapsing part of $(p \restriction \alpha)^\frown\vec{\pi} + t$. More precisely, $\mathbf{Col}(p \restriction \alpha, \vec{\pi}, t) := \varpi_{\ell(p)+k}(p \restriction \alpha^\frown\vec{\pi} + t)$. [17]

Next we show how n-point extension of a condition $(p, c) \in \mathbb{M}$ are formed:

Definition 3.12. Let $(p, c) \in \mathbb{M}$ and $\nu \in A^p_\ell$. Denote $(p, c)^\frown\nu := (p^\frown\nu, c^\frown\nu)$ where $c^\frown\nu$ is the function defined as follows:

(1) $\mathrm{dom}(c^\frown\nu) := \mathrm{dom}(c)$;

(2) $\mathrm{dom}((c^\frown\nu)(\alpha))$ is the collection of all $\langle\langle \pi_{\ell+1}, \dots, \pi_k\rangle, t\rangle$ such that

$$\langle\langle \pi_{\mathrm{mc}(a^p_\ell),\mathrm{mc}(a^p_\ell\cap\alpha)}(\nu)\rangle^\frown\langle \pi_{\ell+1}, \dots, \pi_k\rangle, t\rangle \in \mathrm{dom}(c(\alpha));$$

(3) $(c^\frown\nu)(\alpha, \vec{\pi}, t) := c(\alpha, \langle \pi_{\mathrm{mc}(a^p_\ell),\mathrm{mc}(a^p_\ell\cap\alpha)}(\nu)\rangle^\frown\vec{\pi}, t)$

For $\vec{\nu} \in \prod_{\ell(p)\leq n\leq k} A^p_n$ define $(p, c)^\frown\vec{\nu}$ by recursion on $|\vec{\nu}|$ in the natural way.

There is also a natural way to extend a condition by just strengthening (part of) its collapsing component:

Definition 3.13. Let $(p, c) \in \mathbb{M}$ and $t \preceq \varpi_n(p)$, for $n \leq \ell(p)$. Denote by $(p, c) + t$ the pair $(p + t, c + t)$ where $c + t$ is the function with

(1) $\mathrm{dom}(c + t) := \mathrm{dom}(c)$;

[17] Note that t is an initial piece of these collapses, and if $|t| = k$, then $\mathbf{Col}(p\restriction\alpha, \vec{\pi}, t) = t$.

(2) $\mathrm{dom}((c+t)(\alpha))$ consists of all $\langle \vec{\pi}, s\rangle$, such that for some $k \geq l(p)$,

$\vec{\pi} \in \prod_{\ell(p)\leq n\leq k} \pi_{\mathrm{mc}(a^p_i),\mathrm{mc}(a^p_i\cap\alpha)}``A^p_i \;\wedge\; s \in \bigcup_{n\leq k} \mathbb{S}_n \downarrow \varpi_n(p \restriction \alpha^\frown \vec{\pi} + t)$.

(3) $(c+t)(\alpha, \vec{\pi}, s) := c(\alpha, \vec{\pi}, s)$ for all $\langle \vec{\pi}, s\rangle \in \mathrm{dom}((c+t)(\alpha))$.

Remark 3.14. Above, we simply restrict the original information from c to the relevant domains, so that $c^\frown\vec{\nu}$ (resp. $c+t$) is of the right shape.

Finally, the ordering of $\mathbb{M}$ is defined as follows:

Definition 3.15. For two conditions $(p,c),(q,d) \in \mathbb{M}$ we write $(p,c) \leq (q,d)$ if $(p,c) \leq^* (q,d)^\frown\vec{\nu}$ for some $\vec{\nu} \in \prod_{\ell(p)\leq n\leq \ell(q)} A^q_n$.

3.3. $\mathbb{M}$ **is weak-**$(\Sigma,\vec{\mathbb{S}})$**-Prikry.** Our next aim is to show that $\mathbb{M}$ inherits all the pleasant properties of $\mathbb{P}$, as proved in [PRS21, §4]; namely, property $\mathcal{D}$, the CPP, the existence of *nice projections* ϖ_n, suitability for reflection, etc. To prove all of these properties at once we shall utilize the notion of *supernice forking projection* introduced in [PRS21, §5]. Specifically, we will exhibit a (supernice) forking projection $(⫛, \pi)$ from $\mathbb{M}$ to $\mathbb{P}$ and later appeal to the results in [PRS21, §5] saying that such operations preserve all the relevant properties of $\mathbb{P}$.

Definition 3.16. Let $(⫛, \pi)$ be the pair defined as follows:

(1) $\pi\colon \mathbb{M} \to \mathbb{P}$ is given by $\pi(p,c) := p$;
(2) For each $(p,c) \in \mathbb{M}$ and $q \leq p$ define $⫛(p,c)(q) := (q, c_q)$ where

- $\mathrm{dom}(c_q) = \mathrm{dom}(c)$;
- $\mathrm{dom}(c_q(\alpha))$ is the collection of all $\langle \vec{\pi}, t\rangle$, such that for some $k \geq \ell(q)$,

$\vec{\pi} \in \prod_{\ell(q)\leq n\leq k} \pi_{\mathrm{mc}(a^q_i),\mathrm{mc}(a^q_i\cap\alpha)}``A^q_i \;\wedge\; t \in \bigcup_{n\leq k} \mathbb{S}_n \downarrow \varpi_n(q \restriction \alpha^\frown \vec{\pi})$.

- For each $\langle \vec{\pi}, t\rangle \in \mathrm{dom}(c_q(\alpha))$,

$c_q(\alpha, \vec{\pi}, t) := (c^\frown\vec{\nu})(\alpha, \langle \pi_{\mathrm{mc}(a^q_i\cap\alpha),\mathrm{mc}(a^p_i\cap\alpha)}(\pi_i) \mid \ell(q) \leq i \leq |\vec{\pi}|\rangle, \mathbf{Col}(q{\restriction}\alpha, \vec{\pi}, t))$,

where $\vec{\nu}$ is the unique such that $q \leq^* p^\frown\vec{\nu}$

Remark 3.17. The idea behind c_q is the following. First, we find $\vec{\nu}$ witnessing $q \leq^* p^\frown\vec{\nu}$. After that we take the map $(c^\frown\vec{\nu})(\alpha,\cdot,\cdot)$ and essentially restrict it to a suitable domain for this to be condition with Prikry part q. Thus (q,c_q) is the weakest extension if (p,c) with first coordinate q.

We use $\mathbf{Col}(q \restriction \alpha, \vec{\pi}, t)$ instead of just t to ensure that $(q,c_q) \leq (p,c)$ (see our definition of the forcing order in Definition 3.11).

Also define $\vec{\varsigma} := \langle \varsigma_n \mid n < \omega\rangle$ where $\varsigma_n := \varpi_n \circ \pi$.

We show that $(⫛, \pi)$ is a supernice forking projection [PRS21, Definitions 5.1, 5.2, 5.4].

Lemma 3.18. $(⫛, \pi)$ *defines a forking projection between* $\mathbb{M}$ *and* $\mathbb{P}$.

Moreover, $(⫛, \pi)$ *is super nice from* $(\mathbb{M}, \vec{\varsigma})$ *to* $(\mathbb{P}, \vec{\varpi})$.

Proof. Let us go over the clauses of [PRS21, Definition 5.1].

(1) π is clearly a projection from $\mathbb{M}$ to $\mathbb{P}$.

(2) For each $q \leq p$ the pair $(q, c_q) := \pitchfork(p, c)(q)$ is a condition in $\mathbb{M}$. The only subtle point in the verification is the increasigness requirement in Definition 3.11(3). Thus, suppose that $q^\frown\vec{\sigma} + s \leq q^\frown\vec{\pi} + t$. It can be showed that $(p^\frown\vec{\nu}) \restriction \alpha^\frown\vec{\sigma} + \mathbf{Col}(q \restriction \alpha, \vec{\sigma}, s) \leq (p^\frown\vec{\nu}) \restriction \alpha^\frown\vec{\pi} + \mathbf{Col}(q \restriction \alpha, \vec{\pi}, s)$.[18] Since (p, c) was a condition, by the definition of c_q, we have that the former condition forces "$c_q(\alpha, \vec{\sigma}, s) \leq c_q(\alpha, \vec{\pi}, t)$". Finally

$$q \restriction \alpha^\frown\vec{\sigma} + s \leq (p^\frown\vec{\nu}) \restriction \alpha^\frown\vec{\sigma} + \mathbf{Col}(q \restriction \alpha, \vec{\sigma}, s),$$

hence the left-hand-side condition forces "$c_q(\alpha, \vec{\sigma}, s) \leq c_q(\alpha, \vec{\pi}, t)$".

The fact that $\pitchfork(p, c)(\cdot)$ is a $\leq$-increasing map can be proved similarly playing with the definitions of c_r and c_q.

(3) Given $p \in \mathbb{P}$ its *lift* $\lceil p \rceil^{\mathbb{M}}$ to a condition in $\mathbb{M}$ is simply the pair $(p, \varnothing)$.

(4) Let $a := (p, c)$ and $b := (q, d)$ be such that $b \leq a$. Let $\vec{\nu}$ be the unique such that $q \leq^* p^\frown\vec{\nu}$. It is easy to show that the $\leq$-greatest extension of a above b with the same length as b (denoted by $w(a, b)$ in [PRS21]) is $(p^\frown\vec{\nu}, c^\frown\vec{\nu})$. Similarly, $w(p, q) = p^\frown\vec{\nu}$.

By definition of $\pitchfork(p, c)$ we have that

$$\pitchfork(p, c)(p^\frown\vec{\nu}) := (p^\frown\vec{\nu}, c_{p^\frown\vec{\nu}}).$$

We claim that $(p^\frown\vec{\nu}, c^\frown\vec{\nu}) = (p^\frown\vec{\nu}, c_{p^\frown\vec{\nu}})$. By definition, the domains of $c^\frown\vec{\nu}$ (resp. $c^\frown\vec{\nu}(\alpha)$) and $c_{p^\frown\vec{\nu}}$ (resp. $c_{p^\frown\vec{\nu}}(\alpha)$) are the same. Also, $c_{p^\frown\vec{\nu}}(\alpha, \vec{\pi}, t) := c^\frown\vec{\nu}(\alpha, \vec{\pi}, \mathbf{Col}((p^\frown\vec{\nu}) \restriction \alpha, \vec{\pi}, t))$. Since $\mathbf{Col}((p^\frown\vec{\nu}) \restriction \alpha, \vec{\pi}, t)$ is the collapses of $p^\frown\vec{\nu}^\frown\vec{\pi} + t$, we get that $(p^\frown\vec{\nu}) \restriction \alpha^\frown\vec{\pi} + t$ equals $(p^\frown\vec{\nu}) \restriction \alpha^\frown\vec{\pi} + \mathbf{Col}((p^\frown\vec{\nu}) \restriction \alpha, \vec{\pi}, t)$. So,

$$(c^\frown\vec{\nu})(\alpha, \vec{\pi}, \mathbf{Col}((p^\frown\vec{\nu}) \restriction \alpha, \vec{\pi}, t)) = (c^\frown\vec{\nu})(\alpha, \vec{\pi}, t).$$

Then, by 3.11(3) we have that $(p^\frown\vec{\nu}) \restriction \alpha^\frown\vec{\pi} + t$ forces

$$\text{"}c_{p^\frown\vec{\nu}}(\alpha, \vec{\pi}, t) \leq (c^\frown\vec{\nu})(\alpha, \vec{\pi}, t)\text{" and "}(c^\frown\vec{\nu})(\alpha, \vec{\pi}, t) \leq c_{p^\frown\vec{\nu}}(\alpha, \vec{\pi}, t)\text{"}.$$

By definition of $\dot{\mathrm{Add}}(\kappa^+, 1)$, both conditions are forced to be equal. If we tweak a bit our definition of $\mathbb{M}$ and require that two $\mathbb{P} \restriction \alpha$-names $c(\alpha, \vec{\pi}, t)$ are equal provided their supporting conditions are so (or, perhaps, simply forcing equivalent) we get that $(p^\frown\vec{\nu}, c^\frown\vec{\nu}) = (p^\frown\vec{\nu}, c_{p^\frown\vec{\nu}})$.

(5) Evident.

(6) Easy to verify.

(7) Let $(p, c) \leq^* (q, d)$ and $r \leq^* p$. It is easy to check that $\pitchfork(p, c)(r) \leq^* \pitchfork(q, d)(r)$ because $\vec{\nu}$ in this case is simply $\varnothing$.

This shows that $(\pitchfork, \pi)$ is a forking projection. Let's prove the moreover:

Claim 3.18.1. *For each $n < \omega$, ς_n is a nice projection from $\mathbb{M}_{\geq n}$ to $\mathbb{S}_n$, and for each $k \geq n$, $\varsigma_n \restriction \mathbb{M}_k$ is again a nice projection.*

Therefore, $(\pitchfork, \pi)$ is nice.

[18] This is the reason why we incorporated $\mathbf{Col}(p \restriction \alpha, \vec{\sigma}, s)$ in our definition of the pure-extension ordering $\leq^*$.

Proof of claim. It suffices to prove Clause (4) of being a nice projection. Suppose that $(p', c') \leq (p, c) + s$ for $(p', c'), (p, c) \in \mathbb{M}$ and $s \preceq \varpi_n(p)$ (i.e, s extends part of the collapses of p). We shall construct a condition $(p^*, c^*) \in \mathbb{M}$ such that $(p^*, c^*) \leq^{\varsigma_n} (p, c)$ and $(p', c') = (p^*, c^*) + \varpi_n(p')$. I.e., (p^*, c^*) will have the same collapses as p up to length n which, when fed with the collapses of p', becomes exactly (p', c').

Let us begin taking $p^* \leq^{\varpi_n} p$ such that $p^* + \varpi_n(p') = p'$. Such a p^* exists since ϖ_n is a nice projection. Note that p^* agrees with p' in all the entries except, perhaps, in the collapses. Let $\vec{\nu}$ be such that $p' \leq^* p^\frown\vec{\nu}$. Note that $p^* \leq^* q^\frown\vec{\nu}$ because p' and p^* agree everywhere, except in the collapses.

Define the map c^* as follows:

- $\mathrm{dom}(c^*) := \mathrm{dom}(c')$,
- $\mathrm{dom}(c^*(\alpha))$ consists of all $\langle\vec{\pi}, t\rangle$, such that for some $k \geq \ell(p) = \ell(p')$,

$$\vec{\pi} \in \prod_{\ell(p)\leq i\leq k} \pi_{\mathrm{mc}(a_i^{p'}),\mathrm{mc}(a_i^{p'}\cap\alpha)} ``A_i^{p'},\ t \in \bigcup_{m\leq k} \mathbb{S}_m \downarrow \varpi_m(p^* \restriction \alpha^\frown\vec{\pi}).$$

Finally, $c^*(\alpha, \vec{\pi}, t)$ is defined according to the following three subcases:

<u>If $\langle\vec{\pi}, t\rangle \in \mathrm{dom}(c'(\alpha))$:</u> Then $c^*(\alpha, \vec{\pi}, t) := c'(\alpha, \vec{\pi}, t)$.

<u>If $\langle\vec{\pi}, t\rangle \notin \mathrm{dom}(c'(\alpha))$ but $\alpha \in \mathrm{dom}(c)$:</u>

$$c^*(\alpha, \vec{\pi}, t) := (c^\frown\vec{\nu})(\alpha, \langle\pi_{\mathrm{mc}(a_i^{p'}\cap\alpha),\mathrm{mc}(a_i^{p}\cap\alpha)}(\pi_i) \mid \ell(p) \leq i \leq k\rangle, \mathbf{Col}(p^*\restriction\alpha, \vec{\pi}, t)).$$

<u>Otherwise:</u> $c^*(\alpha, \vec{\pi}, t) := \{\langle\varnothing, \mathbb{1}_{\mathbb{P}\restriction\alpha}\rangle\}$.

Once can check that (p^*, c^*) is a condition and that $(p^*, c^*) \leq^{\varsigma_n} (p, c)$. Also, it is routine that $(p^*, c^*) + \varpi_n(p') = (p', c')$: the equality in the first coordinate follows from our choice of p^*; the equality in the second one follows by shifting $\mathrm{dom}(c^*(\alpha))$ via $\varpi_n(p')$ and noticing that c^* in those pairs equals c'. This completes the proof of the claim. □

The proof of the previous claim actually showed the following stronger fact: If $(p', c') \leq^{\varsigma_n} (p, c) + s$ and p^* is any condition in $\mathbb{P}$ such that $p^* \leq^{\varpi_n} p$ and $p' = p^* + \varpi_n(p')$ then there is $(p^*, c^*) \leq^{\varsigma_n} (p, c)$ such that $(p', c') = (p^*, c^*) + s$. Looking at [PRS21, Definition 5.4] one realizes that this is the same as saying that the pair $(\pitchfork, \pi)$ is super nice. □

Combining Lemma 3.18 with the theory developed in [PRS21, §5] and the properties of Gitik's EBPFC proved across [PRS21, §4] one deduces that $(\mathbb{M}, \ell, \vec{\varsigma})$ is *weak-*$(\Sigma, \vec{\mathbb{S}})$*-Prikry* with property $\mathcal{D}$. More explicitly, $(\mathbb{M}, \ell, \vec{\varsigma})$ satisfies all the axioms of being $(\Sigma, \vec{\mathbb{S}})$-Prikry (see [PRS21, §3]) with possibly the sole exception of the existence of a compatibility function $c\colon \mathbb{M} \to H_{\kappa^+}$.

3.4. Cardinal structure.

Lemma 3.19. $\mathbb{M}$ *forces* "$\kappa = \aleph_\omega$" *and* "$\kappa^+ = \aleph_{\omega+1}$".

Proof. First, it is not hard to prove that $V^{\mathbb{M}}$ and $V^{\mathbb{P}}$ have the same cardinals $\leq\kappa$. Since $\mathbb{P}$ forces "$\kappa = \aleph_\omega$" then so does $\mathbb{M}$. Also, $\mathbb{M}$ preserves κ^+ by a an argument similar to that of Lemma 2.13 (see Lemma 3.15(4) of [PRS21] for details). These two facts combined yield the lemma. $\square$

Lemma 3.20. $\mathbb{M}$ *collapses all* V*-regular cardinals in* (κ^+, λ).

Proof. The proof is very similar to that of Lemma 2.14 so we just sketch the argument. Let $\alpha \in (\kappa^+, \lambda)$ a V-regular cardinal and $G \subseteq \mathbb{M}$ be V-generic. Denote by $\bar{G}$ and $\bar{G}_\alpha$ the $\mathbb{P}$ and $\mathbb{P} \restriction \alpha$-generic filters induced by G. For each $n < \omega$ let $H_n \subseteq \mathbb{S}_n$ be the V-generic induced from $\bar{G}_\alpha$ for the collapsing forcing $\mathbb{S}_n$. The definition of the collapsing map c_α is similar to that of Lemma 2.14 but now it incorporates the collapsing parts.

Formally, c_α equals

$$\bigcup\{c(\alpha, \vec{\nu}\restriction\alpha, t)_{\bar{G}_\alpha} \mid (p, c) \in G,\, p^\frown\vec{\nu} \in \bar{G},\, t \in \textstyle\bigcup_{n \leq \ell(p)+|\vec{\nu}|} H_n \downarrow \varpi_n((p^\frown\vec{\nu})\restriction\alpha)\},$$

where $H_n \downarrow \varpi_n((p^\frown\vec{\nu}) \restriction \alpha)$ stands for the collection of all conditions in the generic H_n below $\varpi_n((p^\frown\vec{\nu}) \restriction \alpha)$. As in Lemma 2.14, $\vec{\nu} \restriction \alpha$ here denotes the αth-projection of $\vec{\nu}$ using the projections associated to $p \in \mathbb{P}$.

Claim 3.20.1. c_α *is well-defined.*

Proof. Let $c(\alpha, \vec{\nu} \restriction \alpha, t)_{\bar{G}_\alpha}$ and $d(\alpha, \vec{\eta} \restriction \alpha, s)_{\bar{G}_\alpha}$ be as before. These objects come together with conditions $(p, c), (q, d) \in G$ as above. Let $(r, e) \leq (p, c), (q, d)$ in G long enough so that its collapsing part $\varpi_{\ell(r)}(r)$ is stronger than the collapsing parts of $(p, c)+t$ and $(q, d)+s$. This is possible because t and s belong to the generic filters H_n and H_m and as a result are compatible. Clearly, $\varpi_{\ell(r)} \in H_{\ell(r)}$. Let $\vec{\sigma}$ and $\vec{\rho}$ be such that $(r, e) \leq^* (p, c)^\frown\vec{\sigma}, (q, d)^\frown\vec{\rho}$. Arguing as in Lemma 2.14 we have that

$$r \restriction \alpha^\frown\vec{\pi} + u \Vdash_{\mathbb{P}\restriction\alpha} e(\alpha, \vec{\pi}, u) \leq c(\alpha, \vec{\nu} \restriction \alpha, \mathbf{Col}(r \restriction \alpha, \vec{\pi}, u))$$

and

$$r \restriction \alpha^\frown\vec{\pi} + u \Vdash_{\mathbb{P}\restriction\alpha} e(\alpha, \vec{\pi}, u) \leq d(\alpha, \vec{\eta} \restriction \alpha, \mathbf{Col}(r \restriction \alpha, \vec{\pi}, u))$$

for all $\langle\vec{\pi}, u\rangle$ in the domain of $e(\alpha)$. In particular, this holds for all the $\vec{\pi}, u$ in the generic Prikry branch. Hence, $c(\alpha, \vec{\nu} \restriction \alpha, t)_{\bar{G}_\alpha} \cup d(\alpha, \vec{\eta} \restriction \alpha, s)_{\bar{G}_\alpha}$ is a function. $\square$

The argument that c_α collapses all V-regular cardinals in (κ^+, α) is the same as in Lemma 2.14. Simply require in the definition of the dense open set D_β that "$c(\alpha, \vec{\pi}, t)_{\bar{G}_\alpha} = f_\beta(\xi)$" for all $\xi < \kappa$ and $\langle\vec{\pi}, t\rangle \in \mathrm{dom}(c(\alpha))$. $\square$

The following can be proved as Lemma 2.19:

Lemma 3.21. $\mathbb{M}$ *is* λ*-Knaster to* $<\lambda$*-Linked.*
In particular, $\mathbb{M}$ *forces "*$\lambda = \aleph_{\omega+2}$*".* $\square$

3.5. **Projections.**

Lemma 3.22. *For each* $\alpha \in \mathcal{R}$ *the map* $(p, c) \mapsto (p \restriction \alpha, c \restriction \alpha)$ *defines a length-preserving projection from* $\mathbb{M}$ *to* $\mathbb{M} \restriction \alpha$.

Proof. The argument is similar to Lemma 2.20, so we only sketch it.

First, $(p \restriction \alpha, c \restriction \alpha)$ is a well-defined condition in $\mathbb{M} \restriction \alpha$. Second, the map is order-preserving by the very definition of the $\leq$-ordering. Finally, suppose that $(q, d) \leq (p \restriction \alpha, c \restriction \alpha)$. Let $r \leq p$ be such that $r \restriction \alpha \leq^* q$. Say $\vec{\nu}$ is the unique such that $r \leq^* p^\frown\vec{\nu}$. Let e be the function with domain $\mathrm{dom}(e) := \mathrm{dom}(d) \cup (\mathrm{dom}(c) \setminus \alpha)$ such that for each $\beta \in \mathrm{dom}(e)$:

- $\mathrm{dom}(e(\beta))$ consists of all $\langle \vec{\pi}, t\rangle$, such that for some $k \geq \ell(r)$,
  $\vec{\pi} \in \prod_{\ell(r)\leq n\leq k} \pi_{\mathrm{mc}(a^r_n),\mathrm{mc}(a^r_n\cap\beta)}``A^r_n$, $t \in \bigcup_{n\leq k} \mathbb{S}_n \downarrow \varpi_n(r \restriction \beta^\frown\vec{\pi})$;
- for each $\langle \vec{\pi}, t\rangle \in \mathrm{dom}(e(\beta))$, $e(\beta, \vec{\pi}, t)$ equals

$$\begin{cases} (c^\frown\vec{\nu})(\beta, \langle \pi_{\mathrm{mc}(a^r_i\cap\beta),\mathrm{mc}(a^p_i\cap\beta)}(\pi_i) \mid \ell(r) \leq i \leq k\rangle, \mathbf{Col}(r \restriction \beta, \vec{\pi}, t)), & \text{if } \beta \notin \mathrm{dom}(d); \\ d(\beta, \langle \pi_{\mathrm{mc}(a^r_i\cap\beta),\mathrm{mc}(a^q_i\cap\beta)}(\pi_i) \mid \ell(r) \leq i \leq k\rangle, \mathbf{Col}(r \restriction \beta, \vec{\pi}, t)), & \text{if } \beta \in \mathrm{dom}(d). \end{cases}$$

Arguing as in Lemma 2.20, (r, e) is a well-defined condition. A key ingredient now is the following. If $(r \restriction \beta)^\frown\vec{\sigma} + s \leq (r \restriction \beta)^\frown\vec{\pi} + t$ then

$$(q \restriction \beta)^\frown\vec{\sigma}' \restriction \beta + \mathbf{Col}(r \restriction \beta, \vec{\sigma}, t) \leq (q \restriction \beta)^\frown\vec{\pi}' \restriction \beta + \mathbf{Col}(r \restriction \beta, \vec{\pi}, t),$$

where $\vec{\sigma}'$ (resp. $\vec{\pi}' \restriction \beta$) stands for the projection of $\vec{\sigma}$ (resp. $\vec{\pi}$) using the $\pi_{\mathrm{mc}(a^r_n\cap\beta),\mathrm{mc}(a^q_n\cap\beta)}$'s. The same inequality above holds replacing q by $p^\frown\vec{\nu}$.

Using this one can show that $e(\beta, \cdot, \cdot)$ complies with requirement (3) of Definition 3.11. Therefore, (r, e) is a condition.

It is obvious that $(r, e) \leq^* (p, c)^\frown\vec{\nu}$ and that $(r \restriction \alpha, e \restriction \alpha) \leq^* (q, d)$. So, indeed, $(p, c) \mapsto (p \restriction \alpha, c \restriction \alpha)$ is a projection from $\mathbb{M}$ to $\mathbb{M} \restriction \alpha$. $\square$

3.6. **Conclusion.** The next theorem gathers the basic properties of $\mathbb{M}$. For the proof of item (3) we need the additional assumption that our cardinals κ_n's and λ are supercompact.

Theorem 3.23 (Main properties of $\mathbb{M}$)**.**

(1) $\mathbb{M}$ is weak-$(\Sigma, \vec{\mathbb{S}})$-Prikry and has property $\mathcal{D}$;
(2) $\mathbb{M}$ forces the following:
   (a) Cardinals $\geq \lambda$ are preserved while those in (κ^+, λ) are collapsed;
   (b) $\kappa = \aleph_\omega$, $\kappa^+ = \aleph_{\omega+1}$ and $\lambda = \aleph_{\omega+2}$;
   (c) $\aleph_\omega$ is strong limit;
   (d) $\mathsf{GCH}_{<\aleph_\omega}$, provided $V \models \mathsf{GCH}_{<\kappa}$;
   (e) $2^{\aleph_\omega} = \aleph_{\omega+2}$, hence the $\mathsf{SCH}_{\aleph_\omega}$ fails.
   (f) $\mathrm{ITP}(\aleph_{\omega+2})$ holds.
(3) $(\mathbb{M}_n, \mathbb{S}_n, \varsigma_n)$ is suitable for reflection with respect to $\langle \sigma_{n-1}, \kappa_{n-1}, \kappa_n, \kappa^+\rangle$.[19]
(4) For each V-regular cardinal $\alpha \in [\kappa^+, \lambda)$ the map $(p, c) \mapsto (p \restriction \alpha, c \restriction \alpha)$ defines a length-preserving projection from $\mathbb{M}$ to $\mathbb{M} \restriction \alpha$.

Proof. (1) Already proved.

(2) Clauses (a) and (b) have been already proved while (c), (d) and (e) can be established as in [PRS21, Theorem 8.8]. We included Clause (f) just

[19]See [PRS21, Definition 2.10] and recall the notations in page 16.

for completeness. The complete argument (in fact, in a more complicated scenario) is given through the forthcoming §7.

(3) As in [PRS21, Lemma 5.18], it suffices to check that $\mathbb{S}_n \times \mathbb{M}_n^{\varsigma_n}$ forces

$$(\star)\ \text{“}|\kappa^+| = \mathrm{cf}(\kappa^+) = \kappa_n = (\kappa_{n-1})^{++}\text{”}.$$

First, by Lemma in [PRS21], $(\star)$ holds in the generic extension by $\mathbb{P}_n$. Second, by Lemma in [PRS21], $\mathbb{P}_n$ is forcing equivalent to a product $\mathbb{T}_n \times \mathbb{P}_n^{\psi_n}$ where the former is κ_n-cc (of size κ_n) and the latter is κ_n-directed-closed. In particular, $\kappa_n = (\kappa_{n-1})^{++}$ holds in $V^{\mathbb{T}_n}$. Recall that by Lemma 2.21, $\mathbb{M}_n^\pi$ in the non-collapsing scenario was κ^+-closed. The same argument works as well in the collapsing scenario. So, by Easton's lemma, $\mathbb{T}_n$ forces $\mathbb{P}_n^{\psi_n} \times \mathbb{M}_n^\pi$ to be κ_n-distributive. Thus, $\mathbb{T}_n \times (\mathbb{P}_n^{\psi_n} \times \mathbb{M}_n^\pi)$ also forces $\kappa_n = (\kappa_{n-1})^{++}$.

On another front, $\mathbb{P}_n \times \mathbb{M}_n^\pi$ projects to $\mathbb{P}_n$ and preserves κ_n. Since $\mathbb{P}_n$ forces $|\kappa^+| = \mathrm{cf}(\kappa^+) = \kappa_n$, it follows that $\mathbb{P}_n \times \mathbb{M}_n^\pi$ forces $|\kappa^+| = \mathrm{cf}(\kappa^+) = \kappa_n$.

Combining these two arguments we have that $\mathbb{P}_n \times \mathbb{M}_n^\pi$ forces $(\star)$. Finally, since $\mathbb{P}_n \times \mathbb{M}_n^\pi$ projects to $\mathbb{S}_n \times \mathbb{M}_n^{\varsigma_n}$ and this latter to $\mathbb{P}_n$ a sandwich argument implies that $(\star)$ holds in the generic extension by $\mathbb{S}_n \times \mathbb{M}_n^{\varsigma_n}$.

(4) This is Lemma 3.22. □

4. Killing a fragile stationary set

In this section we describe a forcing poset which given a weak-$(\Sigma, \vec{\mathbb{S}})$-Prikry forcing $\mathbb{Q}$ (e.g. the MEBPF, $\mathbb{M}$) and a *fragile stationary set* $\dot{T}$ returns a weak-$(\Sigma, \vec{\mathbb{S}})$-Prikry forcing $\mathbb{A} := \mathbb{A}(\mathbb{Q}, \dot{T})$ killing the stationarity of $\dot{T}$ and projecting onto $\mathbb{Q}$. The above-mentioned poset $\mathbb{A}$ will be a variation of the Sharon functor with collapses [PRS21, §6]. The modification we present here will ensure that there are projections from $\mathbb{A}$ onto $\mathbb{A} \restriction \alpha$, for all $\alpha \in \mathcal{R}$ (Recall that $\mathcal{R}$ is the regular cardinals in $[\kappa^+, \lambda)$.) Let $\mathcal{R}' = \mathcal{R} \cap \mathrm{Inac}$.

We present the arguments for a general $\mathbb{Q}$ rather than for $\mathbb{M}$ to be able to use the obtained results at any successor stage of the eventual iteration. We assume that $\mathbb{Q}$ is a stage of an iteration starting with $\mathbb{M}$.

Setup 4. We are given $\langle(\mathbb{Q} \restriction \alpha, \vec{\varpi}) \mid \alpha \in \mathcal{I}\rangle$ and $\langle \cdot \restriction \alpha \mid \alpha \in \mathcal{I}\rangle$ such that:

(1) $\mathcal{I} \subseteq \mathcal{R}' \cup \{\lambda\}$ is a co-bounded in λ, with $\lambda \in \mathcal{I}$;
(2) $(\mathbb{Q}\restriction\alpha, \vec{\varpi})$ is a weak-$(\Sigma, \vec{\mathbb{S}})$-Prikry having property $\mathcal{D}$ and $\mathbb{Q}\restriction\alpha \subseteq H_\lambda$;
(3) $\mathbb{Q} \restriction \alpha$ is λ-Knaster to $<\lambda$-Linked (see Definition 2.15);
(4) $\mathbb{Q} \restriction \alpha$ projects to $\mathbb{M} \restriction \alpha$.
(5) $\cdot \restriction \alpha \colon \mathbb{Q} \to \mathbb{Q} \restriction \alpha$ is a length-preserving projection, where $\mathbb{Q} := \mathbb{Q} \restriction \lambda$.
(6) for each $p \in \mathbb{Q}$, for all large $\alpha < \lambda$, $p = p \restriction \alpha$.

Let $r^\star \in \mathbb{Q}$ and $\dot{T}$ be a $\mathbb{Q}$-name for an $r^\star$-fragile stationary subset of κ^+ (Definition 6.1 in [PRS21]). By definition of fragility, for all $q \leq r^\star$,

$$q \Vdash_{\mathbb{Q}_{\ell(q)}} \text{“}\dot{T}_{\ell(q)} \text{ is nonstationary”}.$$

Thus, for each $n \geq \ell(r^*)$, let a $\mathbb{Q}_n$-name $\dot{C}_n$ for a club subset of κ^+ such that for all $q \leq r^\star$, $q \Vdash_{\mathbb{Q}_{\ell(q)}} \dot{T}_{\ell(q)} \cap \dot{C}_{\ell(q)} = \emptyset$. Using that $\mathbb{Q}$ is λ-cc (Clause (3)) there is an inaccessible cardinal $\delta < \lambda$ such that $\dot{T}$ and $\dot{C}_n$ are, respectively, $\mathbb{Q} \restriction \alpha$ and $(\mathbb{Q} \restriction \alpha)_n$-names for all $\alpha \in \mathcal{I} \setminus \delta$. Below $\mu = (\kappa^+)^V$ [20]

For each $\alpha \in \mathcal{I} \setminus \delta$ define the following binary relation:

$$R \restriction \alpha := \{(\varrho, q) \in \mu \times \mathbb{Q} \restriction \alpha \mid \forall r \leq_{\mathbb{Q}\restriction\alpha} q \restriction \alpha \; (r \Vdash_{(\mathbb{Q}\restriction\alpha)_{\ell(r)}} \varrho \in \dot{C}_{\ell(r)})\}.$$

Notice that $R \restriction \alpha$ is downwards closed; namely, for all $(\varrho, q) \in R \restriction \alpha$ if $q' \leq_{\mathbb{Q}\restriction\alpha} q$ then $(\varrho, q') \in R \restriction \alpha$.

Define

$$\dot{T}^+ := \{(\check{\varrho}, p) \mid (\varrho, p) \in \mu \times P \;\&\; \forall q \leq p \Vdash_{\mathbb{Q}_{\ell(q)}} \check{\varrho} \notin \dot{C}_{\ell(q)}\}.$$

By [PRS22, Lemma 4.6] the trivial condition of $\mathbb{Q}$ forces $\dot{T} \subseteq \dot{T}^+$ (hence $\dot{T}^+$ is stationary) and if $(\varrho, q) \in R \restriction \lambda$ then $q \Vdash_{\mathbb{Q}} \varrho \notin \dot{T}^+$. Also, by λ-ccness of $\mathbb{Q}$, the $\mathbb{Q}$-name $\dot{T}^+$ can be identified with a $\mathbb{Q}\restriction\alpha$-name such that if $(\varrho, q) \in R \restriction \alpha$ then $q \Vdash_{\mathbb{Q}\restriction\alpha} \varrho \notin \dot{T}^+$. By shrinking $\mathcal{I} \setminus \delta$ we ensure this for every $\alpha \in \mathcal{I} \setminus \delta$. Denote $\mathcal{J} := \mathcal{I} \setminus \delta$.

Notation 4.1. Given $p \in \mathbb{Q}$ we denote by A^p_i's the measure one sets of the EBPF-part of p. For $\vec{\nu} \in \prod_{\ell(p) \leq i \leq k} A^p_i$, we define $p^\frown \vec{\nu}$ as in the previous sections, i.e. this is the weakest extension of p whose EBPFpart is given by $\vec{\nu}$. Also denote the corresponding projections by $\pi_{\mathrm{mc}(a^p_i), \mathrm{mc}(a^p_i \cap \alpha)}$.

Definition 4.2. Let $p \in \mathbb{Q}$. A function S is a *$\mathcal{J}$-$\langle p, \vec{\mathbb{S}}\rangle$-labeled tree* if $\mathrm{dom}(S) \in [\mathcal{J}]^{\leq\kappa}$ and for each $\alpha \in \mathcal{J}$ the following hold:

(1) $S(\alpha)\colon \mathrm{dom}(S(\alpha)) \to \{C \subseteq \kappa^+ \mid C \text{ closed and bounded}\}$ where

$\mathrm{dom}(S(\alpha)) := \{\langle \vec{\pi}, t\rangle \mid \vec{\pi} \in \prod_{\ell(p) \leq n \leq k} \pi_{\mathrm{mc}(a^p_i)\,\mathrm{mc}(a^p_i \cap \alpha)}\text{“}A^p_i \wedge t \in \bigcup_{n \leq k} \mathbb{S}_n \downarrow \varpi_n(p\restriction\alpha^\frown\vec{\pi})\}$.

(2) for all $\langle \vec{\pi}, t\rangle \in \mathrm{dom}(S(\alpha))$,

$$p \restriction \alpha^\frown \vec{\pi} + t \Vdash_{\mathbb{Q}\restriction\alpha} S(\alpha, \vec{\pi}, t) \cap \dot{T}^+ = \emptyset.$$

(3) if $\langle \vec{\pi}, t\rangle, \langle \vec{\sigma}, s\rangle \in \mathrm{dom}(S(\alpha))$ and $p \restriction \alpha^\frown \vec{\sigma} + s \leq p \restriction \alpha^\frown \vec{\pi} + t$ then

$$S(\alpha, \vec{\pi}, t) \subseteq S(\alpha, \vec{\sigma}, s).$$

(4) there is $m(S) < \omega$, called the *delay* for S, such that, for each $\alpha \in \mathcal{J}$, if $p \restriction \alpha^\frown \vec{\sigma} + s \leq p \restriction \alpha^\frown \vec{\pi} + t$ and $\ell(p \restriction \alpha^\frown \vec{\pi}) \geq \ell(p \restriction \alpha) + m_\alpha$ then $(\max(S(\alpha, \vec{\sigma}, s)), p \restriction \alpha^\frown \vec{\pi}) \in R \restriction \alpha$.

Remark 4.3. For each $\alpha \in \mathcal{J}$, $S(\alpha, \cdot, \cdot)$ is a $\langle p\restriction\alpha, \vec{\mathbb{S}}\rangle$-labeled tree in the sense of [PRS21, Definition 6.3] with respect the binary relation $R\restriction\alpha$. Thus, the above is just a multidimensional version of the notion considered in [PRS21], with a uniform delay. The common delay will help us define the type map in Definition 4.18. The requirement $\mathrm{dom}(S) \in [\mathcal{J}]^{\leq\kappa}$ is used in Lemma 4.7 and Fact 4.20.

[20] This will remain a cardinal in the final generic extension, but will be collapsed in the orthogonal traces $\mathbb{P}_n$.

Definition 4.4. $\vec{S} = \langle S_i \mid i \leq \gamma^{\vec{S}} \rangle$ is a $\mathcal{J}$-$\langle p, \vec{\mathbb{S}} \rangle$*-strategy* if $\gamma^{\vec{S}} < \kappa^+$, $\mathrm{dom}(S_i) = \mathrm{dom}(S_0)$ for all $i \leq \gamma^{\vec{S}}$, and for each $\alpha \in \mathrm{dom}(S_0)$, $\vec{S}(\alpha) := \langle S_i(\alpha) \mid i \leq \gamma^{\vec{S}} \rangle$ is a $\langle p \restriction \alpha, \vec{\mathbb{S}} \rangle$-strategy in the sense of [PRS21, Definition 6.5].

Definition 4.5. For each $\alpha \in \mathcal{J}$, let $\mathbb{A} \restriction \alpha := \mathbb{A}(\mathbb{Q} \restriction \alpha, \mathcal{J} \cap \alpha^+, \dot{T}^+)$ be the poset consisting of pairs $(p, \vec{S})$ such that the following hold:

(1) $p \in \mathbb{Q} \restriction \alpha$;
(2) $\vec{S}$ is either empty or a $(\mathcal{J} \cap \alpha^+)$-$\langle p \restriction \alpha, \vec{\mathbb{S}} \rangle$-strategy.

Write $(q, \vec{Q}) \leq (p, \vec{S})$ if and only if $q \leq p$, $\gamma^{\vec{Q}} \geq \gamma^{\vec{S}}$, $\mathrm{dom}(S_0) \subseteq \mathrm{dom}(Q_0)$ and for each $i \leq \gamma^{\vec{S}}$, $\beta \in \mathrm{dom}(S_i)$ and $\langle \vec{\pi}, t \rangle \in \mathrm{dom}(Q(\beta))$,

$$Q_i(\beta, \vec{\pi}, t) = S_i(\beta, \langle \pi_{\mathrm{mc}(a_i^q \cap \beta), \mathrm{mc}(a_i^p \cap \beta)}(\pi_i) \mid \ell \leq i \leq \ell + |\vec{\pi}| \rangle, \mathbf{Col}(q \restriction \beta, \vec{\pi}, t)),$$

where $\mathbf{Col}(q \restriction \beta, \vec{\pi}, t)$ is as in Definition 3.11.

Finally, given $(p, \vec{S})$ and $\vec{\nu}$ from the measure one sets of p, define $(p, \vec{S})^\frown \vec{\nu} = (p^\frown \vec{\nu}, \vec{S}') \leq (p, \vec{S})$, where $\mathrm{dom}(S) = \mathrm{dom}(S')$ and $\gamma^{\vec{S}} = \gamma^{\vec{S}'}$. This is the weakest extension of (p, S) with first coordinate $p^\frown \vec{\nu}$.

Our main forcing will be $\mathbb{A} := \mathbb{A} \restriction \lambda$.

4.1. **Projections.**

Lemma 4.6. *For each $\alpha \in \mathcal{J}$ there exists a length-preserving projection $\restriction\alpha \colon \mathbb{A} \to \mathbb{A} \restriction \alpha$ given by $(p, \vec{S}) \mapsto (p \restriction \alpha, \langle S_i \restriction \alpha^+ \mid i \leq \gamma^{\vec{S}} \rangle)$.*

Proof. A moment of reflection makes clear that $\cdot \restriction \alpha$ is well-defined and order-preserving. Suppose $(q, \vec{Q}) \leq (p \restriction \alpha, \vec{S} \restriction \alpha)$ and let $r \leq p$ be such that $r \restriction \alpha \leq^* q$. This $r \in \mathbb{Q}$ exists by (5) in Setup 4. Define $\vec{R} := \langle R_i \mid i \leq \gamma^{\vec{Q}} \rangle$ where each R_i has domain $\mathrm{dom}(Q_i) \cup (\mathrm{dom}(S_i) \setminus \alpha^+)$ and $\mathrm{dom}(R_i(\beta))$ is

$$\{\langle \vec{\pi}, t \rangle \mid \vec{\pi} \in \textstyle\prod_{\ell(r) \leq n \leq k} \pi_{\mathrm{mc}(a_i^r)\, \mathrm{mc}(a_i^r \cap \beta)} \text{“} A_i^r \wedge t \in \bigcup_{n \leq k} \mathbb{S}_n \downarrow \varpi_n(r \restriction \beta^\frown \vec{\pi})\}.$$

For each $\beta \in \mathrm{dom}(R_i)$ and $\langle \vec{\pi}, t \rangle \in \mathrm{dom}(R_i(\beta))$, $R_i(\beta, \vec{\pi}, t)$ equals

$$\begin{cases} S_{\min\{i, \gamma^{\vec{S}}\}}(\beta, \langle \pi_{\mathrm{mc}(a_i^r \cap \beta), \mathrm{mc}(a_i^p \cap \beta)}(\pi_i) \mid \ell(p) \leq i \leq k \rangle, \mathbf{Col}(r \restriction \beta, \vec{\pi}, t)), & \text{if } \beta \geq \alpha^+; \\ Q_i(\beta, \langle \pi_{\mathrm{mc}(a_i^r \cap \beta), \mathrm{mc}(a_i^q \cap \beta)}(\pi_i) \mid \ell(r) \leq i \leq k \rangle, \mathbf{Col}(r \restriction \beta, \vec{\pi}, t)), & \text{if } \beta < \alpha^+. \end{cases}$$

It is routine to check that $(r, \vec{R})$ is a well-defined condition in $\mathbb{A}$ and that $(r, \vec{R}) \leq (p, \vec{S})$ and $(r, \vec{R}) \restriction \alpha \leq^* (q, \vec{Q})$. The key point is that for each $\beta, \beta' \in \mathrm{dom}(R_i)$ the functions $R_i(\beta, \cdot, \cdot)$ and $R_i(\beta', \cdot, \cdot)$ are independent. $\square$

Lemma 4.7. $\mathbb{A} \subseteq H_\lambda$ *and for each $(p, \vec{S}) \in \mathbb{A}$ there are co-boundedly many $\alpha < \lambda$ such that $(p, \vec{S}) \restriction \alpha = (p, \vec{S})$.*

Proof. This is one reason for requiring $\mathrm{dom}(S_i) \in [\mathcal{J}]^{\leq \kappa}$ in the definition of $\mathcal{J}$-$\langle p, \vec{\mathbb{S}} \rangle$-strategy. Simply let some $\alpha < \lambda$ such that $p \restriction \alpha = p$ above $\sup(\mathrm{dom}(S_0)) < \lambda$. It is clear that $(p, \vec{S}) \restriction \alpha = (p, \vec{S})$. $\square$

Let us argue that $\mathbb{A}$ is weak-$(\Sigma,\vec{\mathbb{S}})$-Prikry with property $\mathcal{D}$. The same is true for the trucations $(\mathbb{A}\restriction\alpha,\vec{\varsigma})$ and $(\mathbb{Q}\restriction\alpha,\vec{\varpi})$ modulo obvious changes.

Definition 4.8 (Maps)**.**

(1) $\vec{\varsigma} := \langle \varpi_n\circ\pi \mid n<\omega\rangle$ where $\pi\colon (p,\vec{S})\mapsto p$.

(2) For $(p,\vec{S})\in\mathbb{A}$ and $q\le p$ define $\pitchfork(p,\vec{S})(q) := (q,\vec{Q})$ where:
- $\vec{Q} := \langle Q_i \mid i\le \gamma^{\vec{S}}\rangle$.
- $\mathrm{dom}(Q_i) := \mathrm{dom}(S_i)$ and $\mathrm{dom}(Q_i(\beta))$ consists of pairs $\langle\vec{\pi},t\rangle$, such that for some $k<\omega$,

$$\vec{\pi}\in \prod\nolimits_{\ell(q)\le n\le k}\pi_{\mathrm{mc}(a^q_i)\,\mathrm{mc}(a^q_i\cap\beta)}\text{“}A^q_i \;\wedge\; t\in \bigcup\nolimits_{n\le k}\mathbb{S}_n\downarrow\varpi_n(q\restriction\beta^\frown\vec{\pi}).$$

- The value of $Q_i(\beta,\vec{\pi},t)$ is

$$S_i(\beta,\langle\pi_{\mathrm{mc}(a^q_i\cap\beta),\mathrm{mc}(a^p_i\cap\beta)}(\pi_i)\mid \ell(q)\le i\le k\rangle,\mathbf{Col}(r\restriction\beta,\vec{\pi},t)).$$

The next lemma can be proved exactly as in Lemmas 6.9 and 6.10 from [PRS21] noting that for each $\alpha\in\mathcal{J}$, $\langle S_i(\alpha)\mid i\le\gamma^{\vec{S}}\rangle$ is as in [PRS21].

Lemma 4.9. $(\pitchfork,\pi)$ *defines a forking projection between* $\mathbb{A}$ *and* $\mathbb{Q}$. *Moreover,* $(\pitchfork,\pi)$ *is supernice from* $(\mathbb{A},\vec{\varsigma})$ *to* $(\mathbb{Q},\vec{\varpi})$. $\square$

By the results in [PRS21, §5] we infer:

Corollary 4.10. $(\mathbb{A},\vec{\varsigma})$ *is weak-*$(\Sigma,\vec{\mathbb{S}})$*-Prikry having property* $\mathcal{D}$.

Remark 4.11. The natural modification of the pair $(\pitchfork,\pi)$ yields a super nice forking projection between $(\mathbb{A}\restriction\alpha,\vec{\varsigma})$ and $(\mathbb{Q}\restriction\alpha,\vec{\varpi})$.

4.2. **Chain condition.** In this section we show that $\mathbb{A}\restriction\alpha$ is λ-Knaster to $<\!\lambda$-Linked, hence verifying Clause (3) of Setup 4. For this purpose we need to introduce an auxiliary property which enables *lifting* the λ-Knaster to $<\!\lambda$-Linkedness of $\mathbb{Q}\restriction\alpha$ to $\mathbb{A}\restriction\alpha$.

Notation 4.12. For a condition p in $\mathbb{Q}$ or $\mathbb{A}$,

$$W(p) := \{p^\frown\vec{\nu}\mid \vec{\nu}\in\prod\nolimits_{\ell(p)\le i\le k}A^p_i,\, k<\omega\}.$$

Namely, those are the weakest extensions of p.

Definition 4.13. Assume that $(\pitchfork,\pi)$ is a forking projection from $\mathbb{A}$ to $\mathbb{Q}$. We say that $(\pitchfork,\pi)$ *is a compatibility forking projection* (c-forking projection for short) if the following is true: Let $\mathcal{Y}\in[\mathbb{Q}]^\lambda$ and $\mathfrak{C}_{\mathbb{Q}}\colon \bigcup_{p\in\mathcal{Y}}W(p)\to H_\delta$ be such that

$$\mathfrak{C}_{\mathbb{Q}}(q)=\mathfrak{C}_{\mathbb{Q}}(r)\;\Rightarrow\;\exists s\in\mathbb{Q}\,(s\le^* q,r)).$$

Setting $\mathcal{Z}:=\{a\in\mathcal{X}\mid \pi(a)\in\mathcal{Y}\}$ there is $\delta^*\in[\delta,\lambda)$ regular and a map

$$\mathfrak{C}_{\mathbb{A}}\colon \bigcup\nolimits_{a\in\mathcal{Z}}W(a)\to H_{\delta^*}$$

such that for each $b,b'\in\mathrm{dom}(\mathfrak{C}_{\mathbb{A}})$,

$$(\mathfrak{C}_{\mathbb{A}}(b)=\mathfrak{C}_{\mathbb{A}}(b')\Rightarrow \mathfrak{C}_{\mathbb{Q}}(\pi(b))=\mathfrak{C}_{\mathbb{Q}}(\pi(b'))\;\wedge\;\pitchfork(b)(r)=\pitchfork(b')(r)),$$

for all $r\le^*\pi(b),\pi(b')$.

Lemma 4.14. *If $(\pitchfork, \pi)$ is a c-forking projection and $\mathbb{Q}$ is λ-Knaster to $<\lambda$-Linked poset then so is $\mathbb{A}$.*

Proof. Fix $\mathcal{X} \in [\mathbb{A}]^\lambda$. Since $\mathbb{Q}$ is λ-Knaster to $<\lambda$-Linked there is a set $\mathcal{Y} \in [\pi``\mathcal{X}]^\lambda$ and $\mathfrak{C}_\mathbb{Q}$ as above. Hence, the definition of c-forking projection gives a compatibility map $\mathfrak{C}_\mathbb{A}$ for $\mathbb{Q}$. Note $\mathcal{Z} \in [\mathcal{X}]^\lambda$ and $\mathfrak{C}_\mathbb{A}$ witness together that $\mathbb{A}$ is λ-Knaster to $<\lambda$-Linked (Definition 2.15). $\square$

Fix $\alpha \in \mathcal{J}$. By virtue of our set-up assumptions, $\mathbb{Q} \restriction \alpha$ is λ-Knaster to $<\lambda$-Linked. Besides, there is a forking projection from $\mathbb{A} \restriction \alpha$ to $\mathbb{Q} \restriction \alpha$ (see Remark 4.11). Call it $(\pitchfork^\alpha, \pi^\alpha)$.

Lemma 4.15. *The pair $(\pitchfork, \pi)$ is a a c-forking projection from $\mathbb{A}$ to $\mathbb{Q}$.*
In particular, $\mathbb{A}$ is λ-Knaster to $<\lambda$-Linked.

Proof. Let $\mathcal{X} \in [\mathbb{A}]^\lambda$, $\mathcal{Y} \in [\pi``\mathcal{X}]^\lambda$, and $\mathfrak{C}_\mathbb{Q} \colon \bigcup_{p\in\mathcal{Y}} W(p) \to H_\delta$. Set $\delta^* := \max\{\delta, \kappa^+\}$ and define $\mathfrak{C}_\mathbb{A} \colon \bigcup_{a\in\mathcal{Z}} W(a) \to H_{\delta^*}$ as follows:

$$\mathfrak{C}_\mathbb{A}(q, \vec{Q}) := \langle \mathfrak{C}_\mathbb{Q}(q), \langle i, \alpha, \mathfrak{C}_\mathbb{Q}(r), Q_i(\alpha, \vec{\pi}, \cdot) \mid i \leq \gamma^{\vec{Q}}, \alpha \in \mathrm{dom}(Q_0), r \in W(q{\restriction}\alpha)\rangle\rangle,$$

where $\vec{\pi}$ is such that $r = q \restriction \alpha ^\frown \vec{\pi}$ and $Q_i(\alpha, \vec{\pi}, \cdot)$ is the fiber map ranging over all suitable t's. Let $b = (q, \vec{Q}), b' = (q', \vec{Q}')$ be in $\mathrm{dom}(\mathfrak{C}_\mathbb{A})$ and suppose that $\mathfrak{C}_\mathbb{A}(q, \vec{Q}) = \mathfrak{C}_\mathbb{A}(q', \vec{Q}')$. Clearly, both q and q' have the same $\mathfrak{C}_\mathbb{Q}$-value. Let $r \leq^* q, q'$ witnessing this. The argument that $\pitchfork(b)(r) = \pitchfork(b')(r)$ is the same as that from [Pov20, Lemma 13.2.5(8)] going over all possible coordinates $\alpha \in \mathrm{dom}(Q_0)$ within the Sharon strategies. $\square$

Remark 4.16. The same result holds for the forking projection $(\pitchfork \restriction \alpha, \pi \restriction \alpha)$ for all $\alpha \in \mathcal{J}$. Thus, $\mathbb{A} \restriction \alpha$ is λ-Knaster to $<\lambda$-Linked.

4.3. **Prikry property and killing non-reflecting stationary sets.**

Lemma 4.17. *Forcing with $\mathbb{A}$ kills the stationarity of $\dot{T}^+$.*

Proof. Let $\mathbb{A}^\star \subseteq \mathbb{A}$ consisting of condition $(p, \vec{S})$ with $\lambda \in \mathrm{dom}(S_i)$ for all $i \leq \gamma^{\vec{S}}$. Clearly, $\mathbb{A}^\star$ is $\leq^*$-dense in $\mathbb{A}$. Let $\mathbb{A}^-$ denote the Sharon functor with interleaved collapses as defined in [PRS21, §6] with inputs $\mathbb{Q}$ and T^+. Arguing as in Lemma 4.6, $\mathbb{A}^\star$ projects onto $\mathbb{A}^-$ via $(p, \vec{S}) \mapsto (p, \langle S_i(\lambda) \mid i \leq \gamma^{\vec{S}}\rangle)$. In particular, $\mathbb{A}^\star$ the stationarity of $\dot{T}^+$ as so does $\mathbb{A}^-$. $\square$

Definition 4.18. Let $\mathrm{tp} \colon \mathbb{A} \to {}^{<\kappa^+}\omega$ be defined as follows:

$$\mathrm{tp}(p, \vec{S}) := \langle m(S_i) \mid i \leq \gamma^{\vec{S}}\rangle,$$

Also, define the maximal type, $\mathrm{mtp}(p, \vec{S}) = m(S_{\gamma^{\vec{S}}})$.

The arguments in [PRS21, Lemma 6.14] show that tp is a *nice type* (see [PRS21, Definitions 5.8 and 5.9]). For the reader's convenience we just reproduce the argument showing that the *ring poset*

$$\mathring{\mathbb{A}}_n := \{a \in \mathbb{A}_n \mid \pi(a) \in \mathring{\mathbb{Q}}_n \ \wedge \ \mathrm{mtp}(a) = 0\}$$

is $\leq^{\varsigma_n}$-dense dense in $\mathbb{A}_n^{\varsigma_n}$ (Clause (8) in [PRS21, Definition 5.9]).

Lemma 4.19. *The ring poset* $\mathring{\mathbb{A}}_n := \{a \in \mathbb{A}_n \mid \pi(a) \in \mathring{\mathbb{Q}}_n \ \wedge \ \mathrm{mtp}(a) = \vec{0}\}$ *endowed with the order* $\leq^{\varsigma_n}$ *is dense in* $\mathbb{A}_n^{\varsigma_n}$.

Proof. Let $(p, \vec{S}) \in \mathbb{A}_n$ be arbitrary and $\rho < \kappa^+$ be such that

$$\rho > \sup\{\max(S_i(\alpha, \cdot, \cdot)) \mid \alpha \in \mathrm{dom}(S_0) \wedge \langle \vec{\pi}, t\rangle \in \mathrm{dom}(S_i(\alpha)) \wedge i \leq \gamma^{\vec{S}}\}.$$

Note that this choice is possible because $|\mathrm{dom}(S_0)| \leq \kappa$. The argument in [PRS21, Claim 6.2.1] gives $q \leq^{\vec{\varpi}} p$ and ϱ with $(\varrho, q) \in R \restriction \lambda$.

Claim 4.19.1. *For each* $\alpha \in \mathcal{J}$, *if* $(\varrho, q) \in R \restriction \lambda$ *then* $(\varrho, q \restriction \alpha) \in R \restriction \alpha$.

Proof of claim. Assume $(\varrho, q) \in R \restriction \lambda$. Let $r \leq_{\mathbb{Q}\restriction\alpha} q \restriction \alpha$ be arbitrary and suppose towards a contradiction that $r \nVdash_{(\mathbb{Q}\restriction\alpha)_{\ell(r)}} \varrho \in \dot{C}_{\ell(r)}$. By $\leq_{(\mathbb{Q}\restriction\alpha)_{\ell(r)}}$-extending r we may assume that $r \Vdash_{(\mathbb{Q}\restriction\alpha)_{\ell(r)}} \varrho \notin \dot{C}_{\ell(r)}$. Since $\restriction\alpha$ is a length-preserving projection we find a condition $q' \leq_{\mathbb{Q}} q$ such that $q' \restriction \alpha \leq_{(\mathbb{Q}\restriction\alpha)_{\ell(r)}} r$. Since $(\varrho, q) \in R \restriction \lambda$, $q' \Vdash_{\mathbb{Q}_{\ell(r)}} \varrho \in \dot{C}_{\ell(r)}$, which yields $q' \restriction \alpha \Vdash_{(\mathbb{Q}\restriction\alpha)_{\ell(r)}} \varrho \in \dot{C}_{\ell(r)}$ (because $\dot{C}_{\ell(r)}$ is a $(\mathbb{Q} \restriction \alpha)_{\ell(r)}$-name). This is a contradiction. □

Let $\vec{Q} := \langle Q_i \mid i \leq \gamma^{\vec{S}} + 1\rangle$ be with $\mathrm{dom}(Q_i) := \mathrm{dom}(S_0)$ and for each $\beta \in \mathrm{dom}(S_i)$, $Q_i(\beta)$ has the following set as a domain:

$$\{\langle \vec{\pi}, t\rangle \mid \vec{\pi} \in \textstyle\prod_{\ell(q) \leq n \leq k} \pi_{\mathrm{mc}(a_i^q)\,\mathrm{mc}(a_i^q \cap \beta)} \text{“} A_i^q \wedge t \in \bigcup_{n \leq k} \mathbb{S}_n \downarrow \varpi_n(q \restriction \beta^\frown \vec{\pi})\}.$$

Finally, $Q_i(\beta, \vec{\pi}, t)$ is defined by the following clauses

$$\begin{cases} S_i(\beta, \vec{\pi}, t), & \text{if } i \leq \gamma^{\vec{S}}; \\ S_{\gamma^{\vec{S}}}(\beta, \vec{\pi}, t) \cup \{\varrho\}, & \text{if } i = \gamma^{\vec{S}} + 1 \end{cases}$$

By the previous claim $(q, \vec{Q})$ is a well-defined condition in $\mathring{\mathbb{A}}_n$. Moreover, $(q, \vec{Q}) \leq^{\varsigma_n} (p, \vec{S})$. This accomplishes the proof of density of $\mathring{\mathbb{A}}_n^{\varsigma_n}$. □

Another important fact about the projection π is the following:

Fact 4.20. $\mathring{\mathbb{A}}_n^{\pi}$ *is* κ^+*-directed-closed.*

This fact can be proved along the lines of [PRS19, Lemma 6.15] using that the labeled $\langle p, \vec{\mathbb{S}}\rangle$-trees have domains in $[\mathcal{J}]^{\leq\kappa}$.

Finally, $(⋔, \pi)$ has the weak mixing property by the arguments in [PRS21, Lemma 6.15] at each of the various $\alpha \in \mathrm{dom}(S)$. Thus one obtains:

Lemma 4.21. tp *witnesses that* $(⋔, \pi)$ *has the weak mixing property* (WMP). *In particular,* $\mathbb{A}$ *has the Strong Prikry Property.* □

4.4. **Conclusion.** Putting everything into the same canopy we get:

Theorem 4.22. *Under the assumptions of Setup 4, there is a co-bounded (in* λ*) set* $\mathcal{J} \subseteq \mathcal{I}$, $\lambda \in \mathcal{J}$, *a sequence* $\langle \mathbb{A} \restriction \alpha \mid \alpha \in \mathcal{J}\rangle$ *of* λ*-Knaster to* $<\lambda$*-Linked weak-*$(\Sigma, \vec{\mathbb{S}})$*-Prikry forcings with property* $\mathcal{D}$ *and projections* $\langle \cdot \restriction \alpha \mid \alpha \in \mathcal{J}\rangle$ *such that, for each* $\alpha \in \mathcal{J}$, *the following hold:*

1. *there is a forking projection from* $\mathbb{A} \restriction \alpha$ *to* $\mathbb{M} \restriction \alpha$ *with the* WMP*;*

(2) $\cdot \restriction \alpha$ *is a length-preserving projection;*
(3) *for each* $n < \omega$, $(\mathring{\mathbb{A}} \restriction \alpha)^{\pi}_{n}$ *is* κ^+*-directed-closed;*
(4) $\mathbb{1} \Vdash_{\mathbb{A}\restriction\alpha} \mu = \kappa^+$;
(5) $\mathbb{A} \restriction \alpha$ *is a subset of* H_λ;
(6) $\mathbb{1} \Vdash_{\mathbb{A}\restriction\alpha}$ *"$\dot{T}^+$ is non-stationary".*

5. The main iteration

In this section we define a weak-$(\Sigma, \vec{\mathbb{S}})$-Prikry forcing yielding a generic extension witnessing the Main Theorem. The eventual poset will be a $(\Sigma, \vec{\mathbb{S}})$-Prikry-styled λ-length iteration $\mathbb{P}_\lambda$ with support $\leq \kappa$ which starts with the MEPFC, $\mathbb{M}$. The iteration kills all the non-reflecting stationary subsets of κ^+ invoking the functor $\mathbb{A}(\cdot, \cdot, \cdot)$ of p. 27.

5.1. **The iteration and its projections.** Let $\psi \colon \lambda \to H_\lambda$ be a surjection, such that for each inaccessible $\beta \leq \lambda$, $\psi \restriction \beta$ is a bookkeeping function; namely, $\mathrm{Im}(\psi) = H_\beta$ and the ψ-preimage of each $x \in H_\beta$ is cofinal in β.

We define a λ-length iteration $\mathbb{P}_\lambda$ with support $\leq \kappa$ as follows. The first step is the poset MEPFC, $\mathbb{M}$. The successor step is given by the functor $\mathbb{A}(\cdot, \cdot, \cdot)$ of p. 27, using the bookkeeping function ψ.

Similarly, for each inaccessible $\beta \leq \lambda$, $\mathbb{Q}^\beta$ is defined to be the β-length, $\leq \kappa$ support iteration, where the first step is $\mathbb{M} \restriction \beta$ and the successor step is given by the function from the previous section, using $\psi \restriction \beta$ as a bookkeeping function. Note that $\mathbb{P}_\lambda$ is simply $\mathbb{Q}^\lambda$.

Notation 5.1. For $\alpha \in [1, \beta)$ denote $\mathbb{Q}^\beta_\alpha$ the α^{th}-stage of the iteration $\mathbb{Q}^\beta$. Note that $\mathbb{Q}^\lambda_\alpha = \mathbb{P}_\alpha$ (the α-th stage of the iteration $\mathbb{P}_\lambda$).

In a mild abuse of notation if $p \in \mathbb{Q}^\beta_\alpha$ and $\bar{\alpha} < \alpha$, $p \restriction \bar{\alpha}$ denotes the natural restriction of p to a condition in $\mathbb{Q}^\beta_{\bar{\alpha}}$, the $\bar{\alpha}^{\text{th}}$-stage subiteration of $\mathbb{Q}^\beta_\alpha$.

Lemma 5.2. *For each* $0 < \alpha < \lambda$ *for all large inaccessible* $\beta < \lambda$, *there are length-preserving projections* $\rho^\beta_\alpha \colon \mathbb{P}_\alpha \to \mathbb{Q}^\beta_\alpha$. *Moreover, these projections are trucation-stable; namely,*

$$(\rho) \qquad \forall \bar{\alpha} < \alpha \ (\rho^\beta_\alpha(p) \restriction \bar{\alpha} = \rho^\beta_{\bar{\alpha}}(p \restriction \bar{\alpha})).$$

Proof. By induction on α. The base case $\alpha = 1$ holds for all inaccessible $\beta < \lambda$, since $\mathbb{M}$ projects to $\mathbb{M} \restriction \beta$.

Suppose that the result holds for all $\bar{\alpha} < \alpha$. We distinguish two cases:

► $\alpha = \bar{\alpha} + 1$. By induction, for all large β, we have a projection $\rho : \mathbb{P}_{\bar{\alpha}} \to \mathbb{Q}^\beta_{\bar{\alpha}} = \mathbb{P}_\alpha \restriction \beta$. By Theorem 4.22, for all large β, this induces a projection $\mathbb{A}(\mathbb{P}_{\bar{\alpha}}, .) \to \mathbb{A}(\mathbb{Q}^\beta_{\bar{\alpha}}, .)$. Since $\mathbb{A}(\mathbb{P}_{\bar{\alpha}}, .) = \mathbb{P}_\alpha$ and $\mathbb{A}(\mathbb{Q}^\beta_{\bar{\alpha}}, .) = \mathbb{Q}^\beta_\alpha$, we are done.

► $\alpha < \lambda$ is limit. The for all $\bar{\alpha} < \alpha$, for all large β, we have projections $\mathbb{P}_{\bar{\alpha}} \to \mathbb{Q}^\beta_{\bar{\alpha}}$. Since λ is regular, that means for all large β, for all $\bar{\alpha} < \alpha$, there are projections $\mathbb{P}_{\bar{\alpha}} \to \mathbb{Q}^\beta_{\bar{\alpha}}$. For all such β, define $\rho^\beta_\alpha \colon \mathbb{P}_\alpha \to \mathbb{Q}^\beta_\alpha$ by setting $\rho^\beta_\alpha(p) := \bigcup_{\bar{\alpha}<\alpha} \rho^\beta_{\bar{\alpha}}(p \restriction \bar{\alpha})$.

Using the induction hypothesis one can check that ρ^β_α satisfies (ρ). Thus, $\rho^\beta_\alpha(p)$ is a well-defined condition in $\mathbb{Q}^\beta_\alpha$ for all $p \in \mathbb{P}_\alpha$. It is routine (if a little technical) to check that this is a projection. $\square$

Lemma 5.3. *Suppose that $j\colon V \to M$ is an elementary embedding with* $\mathrm{crit}(j) = \lambda$. *Then, $j(\mathbb{P}_\lambda)$ projects to $\mathbb{P}_\lambda$.*

Proof. For simplicity, let us write $\mathbb{P}$ for $\mathbb{P}_\lambda$. It suffices to show that $j(\mathbb{P})_\lambda$ projects to $\mathbb{P}$ because $j(\mathbb{P})$ projects to $j(\mathbb{P})_\lambda$ via the map $p \mapsto p \restriction \lambda$.

By Lemma 5.2, $\{\beta < \lambda \mid \mathbb{P}_\alpha \text{ projects to } \mathbb{Q}^\beta_\alpha\}$[21] is $\mathcal{U}$-large for all $\alpha \in [1, \lambda)$, where $\mathcal{U}$ denotes the normal measure obtained from j. In particular, for each such α, $j(\mathbb{P})_\alpha$ projects to $j(\mathbb{Q})^\lambda_\alpha = \mathbb{P}_\alpha$ via a map ϑ^λ_α witnessing (ρ) of Lemma 5.2.

Now, for each $p \in j(\mathbb{P})_\lambda$ let $\alpha(p) < \lambda$ be the first ordinal above the support of p. This ordinal exists as $j(\mathbb{P})$ is an iteration with support of size $\leq j(\kappa) = \kappa$. Next, define $\vartheta(p) := \vartheta^\lambda_{\alpha(p)}(p)$ Using equation (ρ) it is routine to check that ϑ defines a projection as needed. $\square$

5.2. **Chain condition.**

In this section we show that our main iteration $\mathbb{P}_\lambda$ is λ-Knaster. For this we should first verify that all the intermediate stages $\mathbb{P}_\alpha$ are λ-Knaster to $<\lambda$-Linked (see Definition 2.15).

Lemma 5.4. *For $1 \leq \alpha < \beta < \lambda$, $(\pitchfork_{\beta,\alpha}, \pi_{\beta,\alpha})$ is a c-forking projection.*

In particular, $\mathbb{P}_\alpha$ is λ-Knaster to $<\lambda$-Linked for all $1 \leq \alpha < \lambda$.

Proof. The second claim will follow from Lemma 2.19 and Remark 4.14 once we establish that $(\pitchfork_{\alpha,1}, \pi_{\alpha,1})$ is a c-forking projection (Definition 4.13).

We proceed by induction on $1 < \beta < \lambda$. By the previous section, the pair $(\pitchfork_{2,1}, \pi_{2,1})$ is a c-forking projection. So, suppose that $\langle(\pitchfork_{\beta,\alpha}, \pi_{\beta,\alpha}) \mid 1 \leq \alpha < \beta < \gamma\rangle$ is a sequence of c-forking projections.

Case $\gamma = \bar{\gamma} + 1$: This case follows by the same arguments as in Lemma 4.15.

Case $\gamma \in \mathrm{acc}(\lambda)$: Fix $\beta \in [1, \gamma)$ and let us show that $(\pitchfork_{\gamma,\beta}, \pi_{\gamma,\beta})$ is a c-forking projection. Fix $\mathcal{X} \in [\mathbb{P}_\gamma]^\lambda$, $\mathcal{Y} \in [\pi_{\gamma,\beta}``\mathcal{X}]^\lambda$ and $\mathfrak{C}_\beta\colon \bigcup_{p \in Y_\beta} W(p) \to H_\delta$. For each $\epsilon \in [\beta, \gamma)$ set $\mathcal{X}_\epsilon := \pi_{\gamma,\epsilon}``\mathcal{X}$. By our induction hypothesis, $(\pitchfork_{\epsilon,\beta}, \pi_{\epsilon,\beta})$ is a c-forking projection. Invoke this property with respect to $(\mathcal{X}_\epsilon, \mathcal{Y}, c_\beta)$ and obtain a map $\mathfrak{C}_\epsilon\colon \bigcup_{p \in \mathcal{Y}_\epsilon} W(p) \to H_{\delta_\epsilon}$, $\delta_\epsilon \geq \epsilon$, witnessing the statement in Definition 4.13, where $\mathcal{Y}_\epsilon = \{p \in \mathcal{X}_\epsilon \mid \pi_{\epsilon,\beta}(p) \in \mathcal{Y}\}$.

Set $\mathcal{Y}_\gamma := \{p \in \mathcal{X} \mid \pi_{\gamma,\beta}(p) \in \mathcal{Y}\}$, $\delta^* := (\sup_{\beta < \epsilon < \gamma} \delta_\epsilon)^+$ and define $\mathfrak{C}_\gamma\colon \bigcup_{p \in \mathcal{Y}_\gamma} W(p) \to H_{\delta^*}$ as $\mathfrak{C}_\gamma(q) := \{\langle \epsilon, \mathfrak{C}_\epsilon(q \restriction \epsilon)\rangle \mid \epsilon \in [\beta, \gamma)\}$. Note that $\mathfrak{C}_\gamma$ is well-defined. Suppose that $\mathfrak{C}_\gamma(q) = \mathfrak{C}_\gamma(q')$. Clearly, $\mathfrak{C}_\beta(q \restriction \beta) = \mathfrak{C}_\beta(q' \restriction \beta)$. Second, fix $r \leq^*_\beta q \restriction \beta, q' \restriction \beta$. By definition,

$$\pitchfork_{\gamma,\beta}(q)(r) = \bigcup_{\beta \leq \epsilon < \gamma} \pitchfork_{\epsilon,\beta}(q \restriction \beta)(r).$$

[21]Once again note that $\mathbb{Q}^\lambda_\alpha = \mathbb{P}_\alpha$.

Since for each $\epsilon \in (\beta,\gamma)$, $\mathfrak{C}_\epsilon(q\restriction\epsilon) = \mathfrak{C}_\epsilon(q'\restriction\epsilon)$ and $\mathfrak{C}_\epsilon$ witnessed that $(\pitchfork_{\epsilon,\beta},\pi_{\epsilon,\beta})$ is a c-forking projection, $\pitchfork_{\epsilon,\beta}(q\restriction\beta)(r) = \pitchfork_{\epsilon,\beta}(q'\restriction\beta)(r)$. Since this is true for every $\epsilon \in [\beta,\gamma)$ we conclude that $\pitchfork_{\gamma,\beta}(q)(r) = \pitchfork_{\gamma,\beta}(q')(r)$. □

Lemma 5.5. *$\mathbb{P}_\lambda$ is λ-Knaster.*

Proof. Let $X \in [\mathbb{P}_\lambda]^\lambda$ and look at $\mathcal{S} := \{\mathrm{cl}(B_p) \mid p \in X\}$.[22] Since $\mathcal{S} \subseteq [\lambda]^{\leq\kappa}$ and λ is inaccessible we can form a Δ-system $\{\mathrm{cl}(B_p) \mid p \in Y\}$ for some $Y \in [X]^\lambda$. Say this has root Δ and set $\chi := \sup(\Delta)$. Clearly $\chi < \lambda$.

By Lemma 5.4, $\mathbb{P}_\chi$ is λ-Knaster to $<\lambda$-Linked. Hence there is a set $Y_\chi \in [\pi_{\lambda,\chi}``Y]^\lambda$ where every two conditions are $\leq^*_\chi$-compatible. It is routine to verify that $Z := \{p \in Y \mid \pi_{\lambda,\chi}(p) \in Y_\chi\}$ is a collection of pairwise compatible conditions.

□

6. The Ineffable Tree Property at double successors

In this section we prove that the *Ineffable Tree Property* (ITP) holds after forcing with the Σ-Prikry-styled iteration of §5.

Definition 6.1. A sequence $\langle d_x \mid x \in \mathcal{P}_\lambda(\theta)\rangle$ is a *thin $\mathcal{P}_\lambda(\theta)$-list* if $d_x \subseteq x$ and $|\{d_x \cap c \mid c \subseteq x\}| < \lambda$ for club many $c \in \mathcal{P}_\lambda(\theta)$.[23]

Given a thin $\mathcal{P}_\lambda(\theta)$-list $d = \langle d_x \mid x \in \mathcal{P}_\lambda(\theta)\rangle$ and $b \subseteq \theta$ one says that:

- b is a *cofinal branch* through d if $\{x \mid d_x = b \cap x\}$ is unbounded in $\mathcal{P}_\lambda(\theta)$.
- b is an *ineffable branch* through d if $\{x \mid d_x = b \cap x\}$ is stationary in $\mathcal{P}_\lambda(\theta)$.

The *Ineffable Tree Property* holds at λ (in symbols, ITP(λ)) if for all regular cardinal $\theta > \lambda$ every thin $\mathcal{P}_\lambda(\theta)$-list d carries an ineffable branch through it.

For simplicity in this section, we present the proof of ITP when the iteration is without the collapses. Then, in the next section, we explain how to modify the proof to fit the case with collapses.

Suppose that λ is a supercompact cardinal. Fix a regular cardinal $\theta > \lambda$ and let $j\colon V \to M$ be an embedding witnessing θ-supercompactness of λ. Hereafter $\mathbb{P}_\lambda$ denotes the Σ-Prikry-styled iteration without collapses as described in §5. Namely, the first step of the iteration is the Mitchell EBPF without collapses (Definition 2.4) and subsequently invokes the functor of §4 without the t-parts.

Let $G \subseteq \mathbb{P}_\lambda$ a generic filter over V and denote by π the projection from $j(\mathbb{P}_\lambda)$ to $\mathbb{P}_\lambda$ (see Lemma 5.3). Let $H \subseteq j(\mathbb{P}_\lambda)/G$ generic over $V[G]$. Since $\mathbb{P}_\lambda \subseteq H_\lambda$, $j``G \subseteq G * H$, and so j lifts in $V[G*H]$ to $j\colon V[G] \to M[G*H]$.

[22] Recall that B_p was the notation used in [PRS22, §3.1] for the support of p. More explicitly, $B_p := \{\gamma + 1 \mid p(\gamma) \neq \emptyset\}$.

[23] According to Jech [Jec73], $\mathcal{C} \subseteq \mathcal{P}_\lambda(\theta)$ is a club if it is closed and unbounded in the following sense: **Closed**: Given a $\subseteq$-increasing sequence $\langle c_\alpha \mid \alpha < \beta\rangle \subseteq \mathcal{C}$ with $\beta < \lambda$, $\bigcup_{\alpha<\beta} c_\beta \in \mathcal{C}$; **Unbounded:** For each $x \in \mathcal{P}_\lambda(\theta)$ there is $c \in \mathcal{C}$ such that $x \subseteq c$.

Suppose that $d = \langle d_x \mid x \in \mathcal{P}_\lambda(\theta)^{V[G]}\rangle$ is a thin $\mathcal{P}_\lambda(\theta)$-list in $V[G]$. By standard arguments, $b := \{\alpha < \theta \mid j(\alpha) \in j(d)_{j``\theta}\}$ is a $V[G]$-ineffable branch through d (see e.g. [HS19, p.5]). Also, $b \in V[G * H]$ and it is $<\lambda$-approximated in $V[G]$; namely, $b \cap x \in V[G]$ for all $x \in \mathcal{P}_\lambda(\theta)^{V[G]}$.

Working over V, let $\dot{b}$ be a $j(\mathbb{P}_\lambda)$-name for the branch b such that

$$\mathbb{1} \Vdash_{j(\mathbb{P}_\lambda)} \text{“}\dot{b} \text{ is ineffable } \wedge \forall x \in \mathcal{P}_\lambda(\theta)^{V[\dot{G}]}(\dot{b} \cap x \in V[\dot{G}])\text{”},$$

where $\dot{G}$ is the standard name for the generic filter of $\mathbb{P}_\lambda$.

For the rest of this section we suppose towards a contradiction that $\dot{b}$ is not forced to be in $V[\dot{G}]$. By λ-ccness of $\mathbb{P}_\lambda$, $\mathcal{P}_\lambda(\theta)^V$ is $\subseteq$-unbounded in $\mathcal{P}_\lambda(\theta)^{V[G]}$, hence we shall be working with x's in the ground model V.

In Definition 3.5 (see also Remark 3.7) we introduced the fusion ordering $\leq^{*,k}$ of the EBPF forcing. The $\leq^{*,k}$-order lifts naturally to $j(\mathbb{P}_\lambda)$ as follows:

Definition 6.2 (Fusion ordering)**.** For each $k < \omega$ and $u, v \in j(\mathbb{P}_\lambda)$,

$$u \leq^{*,k} v \text{ iff } u \leq^* v \text{ and } \mathsf{EBPF}(u) \leq^{*,k} \mathsf{EBPF}(v),$$

where $\mathsf{EBPF}(u)$ and $\mathsf{EBPF}(v)$ denote the EBPF-part of u and v. The ordering $\leq^{*,k,-}$ is defined analogously just requiring $\mathsf{EBPF}(u) \leq^{*,k,-} \mathsf{EBPF}(v)$.

Lemma 6.3. *There is $\bar{n} < \omega$ and $u \in j(\mathbb{P}_\lambda)$ such that for all $\ell(u) \leq k < \omega$, for all $v \leq^* u$ there is $x \in \mathcal{P}_\lambda(\theta)$ such that for all $y \in \mathcal{P}_\lambda(\theta)$ with $x \subseteq y$ there is $w \leq^{*,k,-} v$ in $j(\mathring{\mathbb{P}}_\lambda)$ all of whose $\bar{n}$-extensions decide the value of $\dot{d} \cap y$.*

Proof. The proof is the same as in [HS19, Lemma 4.4] using the following strengthening of the Strong Prikry Property of $j(\mathbb{P}_\lambda)$: Let $u \in j(\mathbb{P}_\lambda)$, $D \subseteq j(\mathbb{P}_\lambda)$ be dense open and $k \geq \ell(u)$. Then, there is $v \leq^{*,k,-} u$ and $\bar{n} < \omega$ such that every $\bar{n}$-extension of v enters D. This strengthening of the SPP is established as follows: First, it holds for $j(\mathbb{M})$ because it holds for the EBPF.[24] Second, the proof of property $\mathcal{D}$ given in [PRS22, Lemma 3.11] shows that a witness for the SPP for $j(\mathbb{P}_\lambda)$ has as a first coordinate a condition witnessing the same fact for $j(\mathbb{P}_1)$. Since the relation $\leq^{*,k,-}$ just depends on what occurs at the first coordinate we are done. □

Definition 6.4. Let $u, v \in j(\mathbb{P}_\lambda)$ be with the same length ℓ and $k \geq \ell$.

Write $u^{\frown+k} \leq v^{\frown+k}$ if the first $(k-\ell)$-many (a, A)-parts of $\mathsf{EBPF}(u)$ and $\mathsf{EBPF}(v)$ are the same and for each $\vec{\nu} \in \prod_{\ell \leq i \leq k} A_i$, $u^\frown\vec{\nu} \leq^* v^\frown\vec{\nu}$.

Remark 6.5. If $u^{\frown+k} \leq v^{\frown+k}$ then $\mathsf{EBPF}(u) \leq^* \mathsf{EBPF}(v)$. However, this is not longer the case for the Mitchell/Sharon-parts of u and v: Let c^u and c^v denote the Mitchell-parts of u and v. From $u^{\frown+k} \leq v^{\frown+k}$ it is easy to show that $\mathrm{dom}(c^u) \supseteq \mathrm{dom}(c^v)$. However, it might well be that

$$c^u(\alpha, \vec{\nu} \restriction \alpha) \not\leq c^v(\alpha, \vec{\nu} \restriction \alpha)$$

[24] It holds as well in the collapsing scenario thanks to our adjustments upon Gitik's original formulation. See **Reason 1** in page 18.

for $\alpha \in \mathrm{dom}(c^v)$ and a short $\vec{\nu} \in \prod A_i$ (i.e, $|\vec{\nu}| < (k-\ell)$). Here $\vec{\nu} \restriction \alpha$ denotes the sequence given by the successive αth-projections of $\vec{\nu}$.

Morally speaking, $u^{\frown +k} \leq v^{\frown +k}$ means that $u \leq v$ holds, modulo some finite error. Nonetheless –forcing-wise– u behaves as a strengthening of v, in the sense that, if $u^{\frown +k} \leq v^{\frown +k}$ then $u \Vdash v \in \dot{G}$. Thus, for any sentence φ in the language of forcing of $j(\mathbb{P}_\lambda)$ if $v \Vdash \varphi$ then $u \Vdash \varphi$. □

Lemma 6.6.

(1) $\langle j(\mathring{\mathbb{P}}_\lambda), {}^{\frown +k}\rangle$ is κ_{k+1}-closed.

(2) If $u^{\frown +k} \leq v^{\frown +k}$ then there is $w \leq^{*,k} v$ such that $w^{\frown +k} \leq u^{\frown +k}$.

Proof. (1) Let $\langle u_\alpha \mid \alpha < \theta < \kappa_{k+1}\rangle$ be a ${}^{\frown +k}$-decreasing sequence in $j(\mathring{\mathbb{P}}_\lambda)$. Let $\vec{\nu} \in \prod_{\ell \leq i \leq k} A_i$ be a sequence in the common measure one sets. By definition, for each $\vec{\nu}$, $\langle u_\alpha {}^\frown \vec{\nu} \mid \alpha < \theta\rangle$ is $\leq^*$-decreasing in $j(\mathring{\mathbb{P}}_\lambda)$. Using the closure of the $\leq^*$-ordering for the MEBPF define a lower bound for the sequence of first coordinates (i.e. the MEBPF parts) of $\langle u_\alpha {}^\frown \vec{\nu} \mid \alpha < \theta\rangle$. Next, replace the first coordinate of $u_\alpha {}^\frown \vec{\nu}$ by this lower bound and use the closure of the ring poset $j(\mathring{\mathbb{P}}_\lambda)$ under κ^+-sequences with the same first coordinate. Let $u_{\vec{\nu}}$ be the resulting bound. Inductively, arrange that $\langle u_{\vec{\nu}} \mid \vec{\nu} \in \prod A_i\rangle$ to be diagonalizable. Then find a condition $u_\theta \leq^{*,k} u_0$ such that $u_\theta {}^\frown \vec{\nu} \leq^* u_{\vec{\nu}}$ for all $\vec{\nu}$. Clearly $u_\theta{}^{\frown +k} \leq u_\alpha{}^{\frown +k}$ for all $\alpha < \theta$.

(2) Let us describe how to define such a w. The EBPF-part of w, p_w, is the same as that of u. The Mitchell-part of w is c_w where: $\mathrm{dom}(c_w) = \mathrm{dom}(c_u)$, for each $\alpha \in \mathrm{dom}(c_w)$ and $\vec{\pi} \in \mathrm{dom}(c_w(\alpha))$, $c_w(\alpha, \vec{\pi}) = c_u(\alpha, \vec{\pi})$ in case $\alpha \notin \mathrm{dom}(c_v)$; otherwise, $c_w(\alpha, \vec{\pi})$ is defined according to the following cases:

$$c_w(\alpha, \vec{\pi}) := \begin{cases} c_u(\alpha, \vec{\pi}), & \text{if } |\vec{\pi}| \geq k - \ell; \\ c_v(\alpha, \langle \pi_{\mathrm{mc}(a_i^u \cap \alpha), \mathrm{mc}(a_i^v \cap \alpha)}(\pi_i) \mid i \leq |\vec{\pi}|\rangle), & \text{otherwise.} \end{cases}$$

The Sharon-part of w is defined by going over the support of the stronger condition u. Let us just describe the case relative to the first coordinate. Let $\vec{S} = \langle S_i \mid i \leq \gamma^{\vec{S}}\rangle$ and $\vec{Q} = \langle Q_i \mid i \leq \gamma^{\vec{Q}}\rangle$ be the first Sharon strategies of u and v, respectively. The Sharon strategy of w is $\vec{R} := \langle R_i \mid i \leq \gamma^{\vec{S}}\rangle$. For each $i \leq \gamma^{\vec{S}}$, $\mathrm{dom}(R_i) = \mathrm{dom}(S_i)$ and for each $\alpha \in \mathrm{dom}(R_i)$ and $\vec{\pi} \in \mathrm{dom}(R_i)$ define $Q_i(\alpha, \vec{\pi})$ as: $R_i(\alpha, \vec{\pi}) := S_i(\alpha, \vec{\pi})$, provided $\alpha \notin \mathrm{dom}(Q_i)$; otherwise, define $R_i(\alpha, \vec{\pi})$ as before replacing c_u (resp. c_v) by S_i (resp. Q_i) whenever $i \leq \gamma^{\vec{Q}}$, or simply using S_i if $i > \gamma^{\vec{Q}}$.

We leave the details that w is as desired to the interested reader. □

Remark 6.7. The previous proof is flexible enough to yield the following variant of Clause (2) above: Suppose that $u \leq^* v$ and $u^\frown \vec{\nu} \leq^* v^\frown \vec{\nu}$ for all $\vec{\nu} \in \prod_{i=\ell(u)}^k A_i^u$, but the first $(k-\ell)$-measure one sets of u and v are not the same (so, possibly $u^{\frown +k} \not\leq v^{\frown +k}$).[25] Then, there is $w \leq^{*,k} v$ such that

$$w^\frown \vec{\nu} = u^\frown \vec{\nu} \text{ for all } \vec{\nu} \in \textstyle\prod_{i=\ell(u)}^k A_i^u.$$

[25]Recall that this latter was required in the definition of ${}^{\frown +k}$ given in Definition 6.4.

The next is the key technical lemma. Fix $\bar{n}$ as in Lemma 6.3, and work below the condition given by that lemma.

Lemma 6.8 (Splitting Lemma). *Let $u \in j(\mathbb{P}_\lambda)$, $k \geq \ell(u) + \bar{n}$ and δ be a regular cardinal such that $2^{\kappa_k} < \delta < \kappa_{k+1}$. Then, there is a sequence of conditions $\langle u_\xi \mid \xi < \delta \rangle$, a set $y \in \mathcal{P}_\lambda(\theta)$ and $\bar{u} \leq^{*,k,-} \pi(u)$ such that:*

(1) $u_\xi \leq^{*,k,-} u$ (i.e, $u_\xi \leq^* u$ and $a_i^{u_\xi} = a_i^u$ for all $\ell(u) \leq i \leq k$);
(2) every $\bar{n}$-extension of u_ξ decides a value for $\dot{b} \cap y$;
(3) for all $\ell(u) \leq i \leq k$, $A_i^{u_\xi} = A_i$ for some constant A_i;
(4) $\bar{u}^{\frown +k} \leq \pi(u_\xi)^{\frown +k}$ for all $\xi < \delta$;
(5) let $\xi \neq \zeta$ and suppose that v and w are $\bar{n}$-extensions of u_ξ and u_ζ, respectively. Then, $\bar{u}$ forces (in the poset $\mathbb{P}_\lambda$) that the values of $\dot{b} \cap y$ decided by v and w are different.

Proof. In the first stage of the proof we construct a sequence of conditions $\langle (u_\xi, v_\xi) \mid \xi < \delta \rangle$ in $j(\mathbb{P}_\lambda)$ and of elementary submodels $\langle \mathcal{M}_\xi \mid 0 < \xi < \delta \rangle$ as follows. Set $u_0 = v_0 := u$ and let $\mathcal{M}_1 \prec H_\chi$ be with $|\mathcal{M}_1| < \delta$, $u_0 \in \mathcal{M}_1$ and $\mathcal{M}_1^{\kappa_k} \subseteq \mathcal{M}_1$. Suppose that $\langle (u_\xi, v_\xi) \mid \xi < \zeta \rangle$ and $\langle \mathcal{M}_\xi \mid 0 < \xi < \zeta \rangle$ have been defined in a way that the v_ξ's are $\leq^{*,k}$-decreasing (i.e., they have the same (a, A)-part as u up to and including k) and

- $u_{\xi+1} \leq^{*,k,-} v_\xi$ for all ξ.
- u_ξ is in the ring poset $j(\mathring{\mathbb{P}}_\lambda)$ and the delay of v_ξ is $\leq k - \ell$.

If ζ is a limit ordinal set $\mathcal{M}_\zeta := \bigcup_{\xi<\zeta} \mathcal{M}_\xi$; otherwise, $\mathcal{M}_\zeta$ is such that

$$u_{\zeta-1} \in \mathcal{M}_\zeta,\ |\mathcal{M}_\zeta| < \delta,\ \mathcal{M}_{\zeta-1} \subseteq \mathcal{M}_\zeta \text{ and } \mathcal{M}_\zeta^{\kappa_k} \subseteq \mathcal{M}_\zeta.$$

Let v be a $\leq^{*,k}$-lower bound for the previous v_ξ. This choice is possible: first, we have enough closure to take a $\leq^{*,k}$-lower bound for the Mitchell-part of v; second, we can modify the v_ξ's to have Mitchell-part this $\leq^{*,k}$-lower bound and use the κ^+-closure of the ring $j(\mathring{\mathbb{P}}_\lambda)$ with respect to sequences of conditions with common Mitchell-part. Moreover, v has delay $k - \ell$.[26]

Let $u_\zeta \leq^{*,k,-} v$ in $j(\mathring{\mathbb{P}}_\lambda)$ all of whose $\bar{n}$-extensions decide $\dot{b} \cap y_\zeta$ where y_ζ is the θ-trace of the model $\mathcal{M}_\zeta$; namely, $y_\zeta := \mathcal{M}_\zeta \cap \theta$. Denote the EBPF/Mitchell-parts of u_ζ by p_ζ and c_ζ, respectively. The EBPF/Mitchell part of v are q and d. Note that $\mathrm{dom}(d) \subseteq \mathrm{dom}(c_\zeta)$. Next, we define v_ζ. Informally, v_ζ is the "amalgamation" of v and u_ζ i.e. it is below v and a certain restriction of it is below u_ζ.

First, let us define the EBPF-part of v_ζ:

$$q_\zeta := \langle f_0^\zeta, \ldots, f_{\ell-1}^\zeta, (a_\ell, A_\ell, f_\ell^\zeta), \ldots, (a_k, A_k, f_k^\zeta), (p_\zeta)_{k+1}, (p_\zeta)_{k+2}, \cdots \rangle$$

where f_i^ζ's are the Cohens of u_ζ, the (a_i, A_i)'s are from the initial condition u and $(p_\zeta)_i$ is the ith-entry of the EBPF-part of u_ζ.

---

[26] Even though the v_ξ's do not have null delay these are uniformly bounded by a fixed integer. This is enough to be able to take lower bounds and get the same delay.

The Mitchell-part of v_ζ, denoted d_ζ, is defined as follows: Let d_ζ be the function with $\mathrm{dom}(d_\zeta) := \mathrm{dom}(c_\zeta)$ and for each $\alpha \in \mathrm{dom}(d_\zeta)$,

$$\mathrm{dom}(d_\zeta(\alpha)) := [\textstyle\prod_{\ell \le i \le t} \pi_{\mathrm{mc}(a_i^{q_\zeta}),\mathrm{mc}(a_i^{q_\zeta} \cap \alpha)} \text{“} A_i^{q_\zeta}]^{<\omega}.$$

Let us say that $\vec{\pi} \in \mathrm{dom}(d_\zeta(\alpha))$ is *good* if $|\vec{\pi}| \ge k - \ell$ and $\vec{\pi} \in \mathrm{dom}(c_\zeta(\alpha))$.

If $\vec{\pi}$ is good, then for all $\alpha \in \mathrm{dom}(d_\zeta)$, define

$$d_\zeta(\alpha, \vec{\pi}) = c_\zeta(\alpha, \vec{\pi})$$

Otherwise, if $\vec{\pi}$ is not good, we split the definition into two cases:

$$d_\zeta(\alpha, \vec{\pi}) := \begin{cases} d(\alpha, \langle \pi_{\mathrm{mc}(a_i^{q_\zeta} \cap \alpha),\mathrm{mc}(a_i^{q} \cap \alpha)}(\pi_i) \mid \ell \le i \le t\rangle), & \text{if } \alpha \in \mathrm{dom}(d); \\ \{\langle \check{\varnothing}, \mathbb{1}_{\mathbb{P}\restriction\alpha}\rangle\}, & \text{otherwise.} \end{cases}$$

Note that the a-part of q_ζ and q are the same up to k so the above projection in the first $(k-\ell)$-coordinates equals $\pi_{\mathrm{mc}(a_i^{q_\zeta}),\mathrm{mc}(a_i^{q} \cap \alpha)}(\pi_i) = \pi_i$.

It is not hard to check that (q_ζ, d_ζ) is a condition in $j(\mathbb{M})$ - the key point being $(q_\zeta \restriction \alpha)^\frown\vec{\pi} = (p_\zeta \restriction \alpha)^\frown\vec{\pi}$ for all good $\vec{\pi}$. By the construction it is also the case that (q_ζ, d_ζ) is $\le^{*,k}$-stronger than the Mitchell-part of v. The pair (q_ζ, d_ζ) will be the eventual $j(\mathbb{M})$-part of the future condition v_ζ.

Let us define the Sharon-part of v_ζ. For each $\vec{\nu} \in \prod_{\ell \le i \le k} A_i$ define

$$u_{\vec{\nu}} := \begin{cases} u_\zeta{}^\frown\vec{\nu}, & \text{if } \vec{\nu} \in \prod_{\ell \le i \le k} A_i^{q_\zeta}; \\ v^\frown\vec{\nu}, & \text{otherwise.} \end{cases}$$

Clearly, $u_{\vec{\nu}} \le^* v^\frown\vec{\nu}$ and the delay of the $u_{\vec{\nu}}$ is exactly 0. For the Sharon-part of u_ζ we take the diagonalization of the above $u_{\vec{\nu}}$'s. More precisely, by the *Weak Mixing lemma* ([PRS22, Lemma 3.10]) there is a condition $v_\zeta \le^{*,k} v$ with $v_\zeta \restriction 1 = (q_\zeta, d_\zeta)$, the delay of v_ζ is $k-\ell$ and $v_\zeta{}^\frown\vec{\nu} \le^* u_{\vec{\nu}}$. This completes the definition of v_ζ and, incidentally, the inductive construction.

After this process we get $\langle (u_\xi, v_\xi) \mid \xi < \delta\rangle$ and $\langle \mathcal{M}_\xi \mid 0 < \xi < \delta\rangle$. Since δ is greater than 2^{κ_k} there is an unbounded set $I \subseteq \delta$ such that for $\xi \in I$, $\langle A_\ell^{u_\xi}, \dots, A_k^{u_\xi}\rangle$ is constant; say, with value $\langle B_\ell, \dots, B_k\rangle$. Unlike in [HS19], the sequence $\langle u_\xi \mid \xi \in I\rangle$ may not be $\le^*$-decreasing because even though $u_{\xi+1} \le^{*,k,-} v_\xi$ the latter is not stronger than u_ξ. However we can argue as follows. Let u_δ be a $\le^{*,k}$-lower bound for the v_ξ's and strengthen it so that its first $(k-\ell)$-many measure one sets are $\langle B_\ell, \dots, B_k\rangle$. Note that for all $\vec{\nu}$ and $\xi \in I$, $u_\delta{}^\frown\vec{\nu} \le^* v_\xi{}^\frown\vec{\nu} \le^* u_\xi{}^\frown\vec{\nu}$. In fact, for each $\vec{\nu}$, $\langle u_\xi{}^\frown\vec{\nu} \mid \xi \in I\rangle$ is decreasing. So, $u_\delta{}^{\frown+k} \le u_\xi{}^{\frown+k}$ for all $\xi \in I$ (recall Definition 6.4).

Let $y = y_\delta := \theta \cap (\bigcup_{\xi<\delta} \mathcal{M}_\xi)$. For each $\xi \in I \cup \{\delta\}$, let X_ξ denote the collection of $\mathbb{P}_\lambda$-names decided to be $\dot{b} \cap y_\xi$ by a $\bar{n}$-extensions of u_ξ.[27] By passing from δ to the unbounded set I, re-enumerate $\langle u_\xi \mid \xi \in I\rangle$ by $\langle u_\xi \mid \xi < \delta\rangle$. Let us summarize the properties of $\langle u_\xi \mid \xi < \delta\rangle$:

[27] Using that $u_\delta{}^{\frown+k} \le u_\xi{}^{\frown+k}$ for all $\xi \in I$ one can show that any $\bar{n}$-extension of u_δ decides the value of $\dot{b} \cap y_\delta$. Thus, it is meaningful to consider X_δ.

(1) For all $\xi < \delta$, each $\bar{n}$-step extension of u_ξ decides the value of $\dot{b} \cap y_\xi$ and X_ξ is the collection of these values; $y_\xi = \mathcal{M}_\xi \cap \theta$ and $u_\xi \in \mathcal{M}_{\xi+1}$
(2) For all ξ the measure one sets of u_ξ at coordinates $[\ell, k]$ are constant with value $\langle B_\ell, \dots, B_k \rangle$; their a-parts at these coordinates are also constant (and equal to the ones from the original condition u).
(3) For all $\vec{\nu} \in \prod_{l \leq i \leq k} B_i$, $\langle u_\xi {}^\frown \vec{\nu} \mid \xi < \delta \rangle$ is $\leq^*$-decreasing.

Claim 6.8.1. *For each $\tau \in X_\eta$ and $\xi \leq \eta$, $\tau \cap y_\xi \in X_\xi$. In addition, there is $\xi^* < \delta$ such that if τ and σ are distinct members of X_δ then there is $\xi < \xi^*$ such that $\pi(u_\xi) \Vdash_{\mathbb{P}_\lambda} \tau \cap y_\xi \neq \sigma \cap y_\xi$.*

Proof of claim. Let $\xi \leq \eta$, and $\tau \in X_\eta$. Let $\vec{\nu}$ be such that $u_\eta {}^\frown \vec{\nu}$ forces $\dot{b} \cap y_\eta = \tau$, and let $\sigma \in X_\xi$ be such that $u_\xi {}^\frown \vec{\nu}$ forces $\dot{b} \cap y_\xi = \sigma$. Then, since $u_\eta {}^\frown \vec{\nu} \leq^* u_\xi {}^\frown \vec{\nu}$, we have that $u_\eta {}^\frown \vec{\nu} \Vdash_{j(\mathbb{P}_\lambda)} \dot{b} \cap y_\xi = \sigma$, and so $\tau \cap y_\xi = \sigma \in X_\xi$.
The second claim follows from $|X_\delta| < \delta$. $\square$

Roughly speaking the above claim says that no more splittings are forced past level ξ^*. The next claim plays the role of the usual branching lemma:

Claim 6.8.2. *For every ordinal $\xi^* < \xi < \delta$ and $\bar{r} \leq \pi(u_\xi)$ there is a set $z \in \mathcal{M}_{\xi+1}$ and a condition $r \in j(\mathbb{P}_\lambda)$ as follows:*

(1) $y_\xi \subseteq z \subseteq y_{\xi+1}$;
(2) $r \leq u_\xi$ and $\pi(r) \leq \bar{r}$;
(3) $r \Vdash_{j(\mathbb{P}_\lambda)}$ "$\dot{b} \cap z = \tau$" but $\pi(u_\xi) \Vdash_{\mathbb{P}_\lambda}$ "$\tau \neq \sigma \cap z$" for all $\sigma \in X_{\xi+1}$.

Proof of claim. Assume otherwise and let ξ and $\bar{r}$ be counter-examples. Let us work in an extension of $\mathcal{M}_{\xi+1}$ by a V-generic $\bar{G} \subseteq \mathbb{P}_\lambda$ with $\bar{r} \in \bar{G}$. Set

$$d = \{\tau_{\bar{G}} \mid \exists z \supseteq y_\xi \, \exists r \in j(\mathbb{P}_\lambda)/\bar{G} \; (r \leq u_\xi \, \wedge \, \pi(r) \leq \bar{r} \, \wedge \, r \Vdash_{j(\mathbb{P}_\lambda)} \dot{b} \cap z = \tau)\}.$$

Since $\dot{b}_G \notin V[\bar{G}]$, d cannot be a branch. So, there is $z \in \mathcal{M}_{\xi+1}$ and $r_1, r_2 \in j(\mathbb{P}_\lambda)/\bar{G}$ with $r_1, r_2 \leq u_\xi$ and $\pi(r_1), \pi(r_2) \leq \bar{r}$, and τ_1, τ_2, τ such that

- $r_i \Vdash \dot{b} \cap z = \tau_i$, for $i = 1, 2$.
- $r_i \Vdash \dot{b} \cap y_\xi = \tau$, for $i = 1, 2$.
- $(\tau_1)_{\bar{G}} \neq (\tau_2)_{\bar{G}}$.

Since we assume that (3) fails there are $\sigma_1, \sigma_2 \in X_{\xi+1}$ such that $(\tau_i)_{\bar{G}} = (\sigma_i)_{\bar{G}} \cap z$ for $i = 1, 2$. So, $(\sigma_1)_{\bar{G}} \neq (\sigma_2)_{\bar{G}}$ but $(\sigma_1)_{\bar{G}} \cap y_\xi = (\sigma_2)_{\bar{G}} \cap y_\xi = \tau$. This contradicts Claim 6.8.1 above. $\square$

The proof of the above shows that, for each $\bar{r} \leq \pi(u_\xi)$, there is a dense set of conditions r below the *meet* of u_ξ and $\bar{r}$ which satisfy the conclusion of the claim; call it $D_{\xi,\bar{r}}$. More formally, for every $w \leq u_\xi$ such that $\pi(w) \leq \bar{r}$ there is $w' \leq w$ in $D_{\xi,\bar{r}}$. Combining this with Strong Prikry lemma, we get:

Claim 6.8.3. *For every $\xi^* < \xi < \delta$ and $\bar{r} \leq^{*,k,-} \pi(u_\xi)$ there is $r \leq^{*,k,-} u_\xi$ with $\pi(r) \leq^{*,k,-} \bar{r}$ and a set $z \in \mathcal{M}_{\xi+1}$, such that*

(1) $y_\xi \subseteq z \subseteq y_{\xi+1}$;

(2) every $\bar{n}$-step extension of r forces that $\dot{b} \cap z = \tau$ for some τ, but $\pi(u_\xi) \Vdash_{\mathbb{P}_\lambda}$ "$\tau \neq \sigma \cap z$" for all $\sigma \in X_{\xi+1}$.

Proof of claim. Let $r \leq^{*,k,-} u_\xi$ with $\pi(r) \leq^{*,k,-} \bar{r}$ be such that for some $n \geq \bar{n}$, every n-step extension of r is in $D_{\xi,\bar{r}}$. For all n-step extension $r^\frown \vec{\nu}$, let $z_{\vec{\nu}}$ witness membership in $D_{\xi,\bar{r}}$ and let $z = \bigcup_{\vec{\nu}} z_{\vec{\nu}}$. Let $r' \leq^{*,k,-} r$ be such that every $\bar{n}$-step extension of r' decides $\dot{b} \cap z$. Then r' is as desired. $\square$

For simplicity, say $\xi^* = 0$. Let us define a sequence $\langle r_\xi, s_\xi \mid \xi < \delta\rangle$ by induction as follows. Let $\bar{r} \leq^{*,k} \pi(u_0)$ be such that $\bar{r}^{\frown +k} \leq \pi(u_\delta)^{\frown +k}$ (see Lemma 6.6(2)). Let $r_0 \leq^{*,k,-} u_0$ be as in the above claim when regarded for $\xi = 0$ and $\bar{r}$. By further $\leq^{*,k,-}$-extending r_0 we may assume that all of its $\bar{n}$-extensions decide $\dot{b} \cap y$. Let $s_0 \leq^{*,k} \bar{r}$ be such that

$$s_0{}^\frown \vec{\nu} = \pi(r_0)^\frown \vec{\nu} \text{ for all } \vec{\nu} \in \prod_{i=\ell}^k A_i^{\pi(r_0)}.$$

To ensure this choice we make use of Remark 6.7 in page 35.

Suppose that $\langle r_\zeta, s_\zeta \mid \zeta < \xi\rangle$ has been defined so that

- $\vec{s} := \langle s_\zeta \mid \zeta < \xi\rangle$ is $\leq^{*,k}$-decreasing;
- $\pi(r_\zeta)^\frown \vec{\nu} = s_\zeta{}^\frown \vec{\nu}$ for all $\vec{\nu} \in \prod_{i=\ell}^k A_i^{\pi(r_\zeta)}$ and $\zeta < \xi$.

Since $\xi < \delta < \kappa_{k+1}$ there is a $\leq^{*,k}$-lower bound for $\vec{s}$; call it s^*. By Lemma 6.6(2) there is $\bar{r}_\xi \leq^{*,k} \pi(u_\xi)$ such that $\bar{r}_\xi{}^{\frown +k} \leq s^{*\frown +k}$. Apply the previous claim with respect to ξ and $\bar{r}_\xi$ to produce $r_\xi \leq^{*,k,-} u_\xi$ such that $\pi(r_\xi) \leq^{*,k,-} \bar{r}_\xi$. As before, we may assume that all the $\bar{n}$-extensions of r_ξ decide $\dot{b} \cap y$. Using Remark 6.7, let $s_\xi \leq^{*,k} s^*$ as above.

The upshot of the previous construction is a sequence $\langle r_\xi \mid \xi < \delta\rangle$ such that $\langle \pi(r_\xi)^\frown \vec{\nu} \mid \xi < \delta\rangle$ is $\leq^*$-decreasing, provided $\vec{\nu}$ is a Prikry point common to the first $(k-\ell)$-many measure one sets of the r_ξ's. By passing to an unbounded subset of δ, we may assume that the measure one sets of the r_ξ's at coordinates ℓ, ..., k are in fact constant. Thus, by construction, $\langle \pi(r_\xi) \mid \xi < \delta\rangle$ is ${}^{\frown +k} \leq$-decreasing so we may take a ${}^{\frown +k} \leq$-lower bound, $\bar{u}$ (Lemma 6.6(1)). Note that $\bar{u}^{\frown +k} \leq \pi(u_\delta)^{\frown +k}$.

Claim 6.8.4. *$\langle r_\xi \mid \xi < \delta\rangle$ and $\bar{u}$ are as in the splitting lemma.*

Proof of claim. It is enough to prove Clause (4) of the lemma. Let $\xi < \zeta$ and s, s' be $\bar{n}$-extensions of r_ξ and r_ζ. Let τ and τ' be the $\mathbb{P}_\lambda$-values of $\dot{b} \cap y$ decided by s and s', respectively. By construction of r_ξ, τ is incompatible with members of $X_{\xi+1}$ as forced by $\pi(u_\xi)$ (by Clause (3) in Claim 6.8.2) Namely, for all $\sigma \in X_{\xi+1}$,

$$\pi(u_\xi) \Vdash_{\mathbb{P}_\lambda} \tau \cap y_{\xi+1} \neq \sigma.$$

On the other hand, since $s' \leq^{\bar{n}} r_\zeta \leq^* u_\zeta$, we have that $\tau' \in X_\zeta$. Hence, by Claim 6.8.1, $\tau' \cap y_{\xi+1} \in X_{\xi+1}$. So,

$$\pi(u_\xi) \Vdash_{\mathbb{P}_\lambda} \tau \cap y_{\xi+1} \neq (\tau' \cap y_{\xi+1}).$$

Since $\bar{u}^{\frown +k} \leq \pi(u_\xi)^{\frown +k}$ we get that $\bar{u}$ also $\mathbb{P}_\lambda$-forces "$\tau \neq \tau'$". $\square$

The above completes the proof of the splitting lemma. □

Remark 6.9. The same argument as above works working below any $\bar{v} \leq^{*,k,-} \pi(u)$. In that case the Splitting Lemma gives a condition $\bar{u} \leq^{*,k,-} \bar{v}$.

Theorem 6.10. ITP(λ) *holds in a generic extension by* $\mathbb{P}_\lambda$.

Proof. For simplicity, suppose the initial condition $u \in j(\mathbb{P}_\lambda)$ has length 0. Let us fix a sequence of measurable cardinals $\langle \delta_n \mid n < \omega \rangle$ such that $2^{\kappa_{\bar{n}+n}} < \delta_n < \kappa_{\bar{n}+n+1}$; say this is witnessed by δ_n-complete measures $\langle \mathcal{U}_n \mid n < \omega \rangle$.

We define a tree of conditions $\langle r_{\vec{\sigma}} \mid \vec{\sigma} \in \prod_{n<m} Y_m,\, m < \omega \rangle \subseteq j(\mathbb{P}_\lambda)$, where each $Y_m \in \mathcal{U}_m$, a $\leq^*$-decreasing sequence $\langle \bar{u}_n \mid n < \omega \rangle \subseteq \mathbb{P}_\lambda$ and an $\subseteq$-increasing sequence $\langle y_n \mid n < \omega \rangle$ in $\mathcal{P}_\lambda(\theta)$ with the following properties:

(1) $r_{\vec{\sigma}} \leq^* r_{\vec{\tau}}$ in case $\vec{\tau} \sqsubseteq \vec{\sigma}$;
(2) $\bar{u}_{n+1} \leq^{*,n,-} \bar{u}_n$;
(3) For each $\vec{\sigma} \in \prod_{n<m} Y_m$ and $\xi \in Y_m$, $r_{\vec{\sigma}^\frown\langle \xi \rangle} \leq^{*,m,-} r_{\vec{\sigma}}$;
(4) For each $\vec{\sigma}$ all the $\bar{n}$-extensions of $u_{\vec{\sigma}}$ decide $\dot{b} \cap y_{|\vec{\sigma}|}$;
(5) For incompatible $\vec{\sigma}$ and $\vec{\tau}$, any two $\bar{n}$-extensions of $u_{\vec{\sigma}}$ and $u_{\vec{\tau}}$ decide incompatible values for $\dot{b} \cap y_{|\vec{\sigma}|}$ and $\dot{b} \cap y_{|\vec{\tau}|}$, as $\mathbb{P}_\lambda$-forced by $\bar{u}_{|\vec{\sigma} \cap \vec{\tau}|}$,

By induction on $|\vec{\sigma}|$. First, apply the Splitting Lemma (Lemma 6.8) to the triple $\langle u, \delta_{\bar{n}}, \pi(u) \rangle$ to obtain $\langle r_\xi \mid \xi < \delta_{\bar{n}} \rangle$ and $\bar{u}_{\bar{n}}$, so that:

- $r_\xi \leq^{*,\bar{n},-} u$ for all $\xi < \delta_0$;
- $\bar{u}_0{}^{\curvearrowright+\bar{n}} \leq \pi(r_\xi)^{\curvearrowright+\bar{n}}$ for all $\xi < \delta_0$;
- $\bar{u}_0 \leq^{*,\bar{n},-} \pi(u)$.

Since $\bar{u}_0{}^{\curvearrowright+\bar{n}} \leq \pi(r_0)^{\curvearrowright+\bar{n}}$, there is $w \leq^{*,\bar{n}} \pi(r_0)$ such that $w^{\curvearrowright+\bar{n}} \leq \bar{u}_0{}^{\curvearrowright+\bar{n}}$ (by Lemma 6.6). Apply the Splitting Lemma to $\langle r_0, \delta_1, w \rangle$ and get in return a sequence $\langle r_{0,\eta} \mid \eta < \delta_1 \rangle$ and $\bar{u}_1(0)$. As before, the following hold:

- $r_{0,\eta} \leq^{*,\bar{n}+1,-} r_0$ for all $\eta < \delta_1$;
- $\bar{u}_1(0)^{\curvearrowright+\bar{n}+1} \leq \pi(r_{0,\eta})^{\curvearrowright+\bar{n}+1}$ for all $\eta < \delta_1$;
- $\bar{u}_1(0) \leq^{*,\bar{n}+1,-} w$.

We keep applying the Splitting lemma for r_ξ, $\xi < \delta_0$ by induction of ξ. Suppose we have defined $\langle \langle \langle r_{\zeta,\eta} \mid \eta < \delta_1 \rangle, \bar{u}_1(\zeta) \rangle \mid \zeta < \xi \rangle$, so that $\langle \bar{u}_1(\zeta) \mid \zeta < \xi \rangle$ is $\leq^*$-decreasing except for the first $\bar{n}+1$ measure one sets; more formally, $\langle u_1(\zeta)^\frown\vec{\nu} \mid \zeta < \xi \rangle$ is $\leq^*$-decreasing provided $\vec{\nu} \in \bigcap_{\zeta<\xi} \prod_{i\leq\bar{n}} A_i^{\bar{u}_1(\zeta)}$.

▶ If $\xi = \zeta+1$, mimicking the argument in Lemma 6.6(2), find an auxiliary condition $z \leq^{*,\bar{n},-} \pi(r_\xi)$ with $z^{\curvearrowright+\bar{n}} = \bar{u}_1(0)^{\curvearrowright+\bar{n}}$ (i.e., $z^{\curvearrowright+\bar{n}} \leq \bar{u}_1(0)^{\curvearrowright+\bar{n}}$ and $\bar{u}_1(0)^{\curvearrowright+\bar{n}} \leq z^{\curvearrowright+\bar{n}}$). Invoke the Splitting Lemma for $\langle r_\xi, \delta_1, z \rangle$.

▶ ξ is limit. Again, first we need an auxiliary condition. For ease of notation, for the rest of the construction we assume that $\bar{n} = 0$. The general case is analogous.

Claim 6.10.1. *There is a condition* w *such that* $w \leq^{*,\bar{n}} \pi(r_\xi)$ *and* $w^\frown\nu \leq^* \bar{u}_1(\zeta)^\frown\nu$ *for all* $\nu \in A_0^{\bar{u}_1(\zeta)}$ *and* $\zeta < \xi$.

Proof. EBPF-part: The EBPF-parts of $u_1(\zeta)$ (for $\zeta < \xi$) and $\pi(r_\xi)$ take the form

$$\langle (a_0, A_0^\zeta, f_0^\zeta)\rangle^\frown p_\zeta \restriction [1,\omega),$$
$$\langle (a_0, A_0, f_0)\rangle^\frown p_{\pi(r_\xi)} \restriction [1,\omega),$$

where $A_0^\zeta \subseteq A_0$ and $f_0 \subseteq f_0^\zeta$ (because $\bar{u}_0{}^{\frown+0} \leq \pi(r_\xi)^{\frown+0}$).

The EBPF-part of w is defined as

$$p_w := \langle (a_0, A_0, \textstyle\bigcup_{\zeta<\xi} f_0^\zeta)\rangle^\frown \textstyle\bigwedge_{\zeta<\xi}(p_\zeta \restriction [1,\omega)),$$

where the latter is just a lower bound for the displayed conditions.

Mitchell-part: Since $\langle \bar{u}_1(\zeta)^\frown \nu \mid \zeta < \xi\rangle$ is $\leq^*$-decreasing we have

$$\mathrm{dom}(c_\zeta) = \mathrm{dom}(c_\zeta{}^\frown \nu) \supseteq \mathrm{dom}(c_{\bar\zeta}{}^\frown \nu) = \mathrm{dom}(c_{\bar\zeta}) \supseteq \mathrm{dom}(c^{\pi(r_\xi)}),$$

where c_ζ is the Mitchell-part of $\bar{u}_1(\zeta)$. Let $\mathrm{dom}(c_w) := \bigcup_{\zeta<\xi} \mathrm{dom}(c_\zeta)$.

For each $\alpha \in \mathrm{dom}(c_w)$ we choose $\mathrm{dom}(c_w(\alpha))$ in the obvious way towards ensuring $(p_w, c_w) \in \mathbb{M}$. For each $\vec{\pi} \in \mathrm{dom}(c_w)$ define $c_w(\alpha, \vec{\pi})$ as:

▶ If $\alpha \notin \mathrm{dom}(c^{\pi(r_\xi)})$ then

$$c_w(\alpha, \vec{\pi}) := \begin{cases} \bigwedge\{c_\zeta(\alpha, \vec{\sigma}_\zeta) \mid \alpha \in \mathrm{dom}(c_\zeta), \langle \pi_0\rangle \in \mathrm{dom}(c_\zeta(\alpha))\}, & \text{if } |\vec{\pi}| \geq 2; \\ \{\langle \varnothing, \mathbb{1}_{\mathbb{P}\restriction\alpha}\rangle\}, & \text{otherwise,} \end{cases}$$

where $\vec{\sigma}_\zeta$ is the sequence of the projections of $\vec{\pi}$ under $\pi_{\mathrm{mc}(a^w\cap\alpha),\mathrm{mc}(a^{\bar{u}_1(\zeta)}\cap\alpha)}$'s.

▶ If $\alpha \in \mathrm{dom}(c^{\pi(r_\xi)})$ then $c_w(\alpha, \vec{\pi})$ is defined as above replacing the $\mathbb{P}\restriction\alpha$-name in the second case by $c^{\pi(r_\xi)}(\alpha, \langle\pi\rangle)$.

We leave to the reader checking that $(p_w, c_w) \in \mathbb{M}$ and $(p_w, c_w) \leq^{*,0} (p_{\pi(r_\xi)}, c^{\pi(r_\xi)})$. Also note that $(p_w, c_w)^\frown \nu \leq^* (p_\zeta, c_\zeta)^\frown \nu$ for all $\nu \in A_0^\zeta$.

Sharon part: Let us describe what to do at the first Sharon-like strategy. For each $\nu \in A_0$ set $I_\nu := \{\zeta < \xi \mid \nu \in A_0^\zeta\}$. Let w_ν denote a $\leq^*$-lower bound for $\langle \pi(r_\xi)^\frown \nu\rangle^\frown \langle \bar{u}_1(\zeta)^\frown \nu \mid \zeta \in I_\nu\rangle$ (this is possible as $\xi < \delta_0 < \kappa_1$).

Next, diagonalize the sequence $\langle w_\nu \mid \nu \in A_0\rangle$ thus finding $w \leq^{*,0} \pi(r_\xi)$ (in fact, the first coordinate of w is (p_w, c_w)) such that $w^\frown \nu \leq^* w_\nu$ for all $\nu \in A_0$. Note that for each $\zeta < \xi$ and $\nu \in A_0^\zeta$, $w^\frown \nu \leq^* \bar{u}_1(\zeta)^\frown \nu$. □

After this we get an auxiliary condition w such that $w \leq^{*,0} \pi(r_\xi)$ and $w^\frown \nu \leq^* \bar{u}_1(\zeta)^\frown \nu$ for $\nu \in A_0^\zeta$, $\zeta < \xi$. Apply the Splitting Lemma to the triple $\langle r_\xi, \delta_1, w\rangle$ and obtain $\langle r_{\xi,\eta} \mid \eta < \delta_1\rangle$ and $\bar{u}_1(\xi)$. Clearly,

$$\bar{u}_1(\xi)^\frown \nu \leq^* \bar{u}_1(\zeta)^\frown \nu \text{ for all } \nu \in \textstyle\bigcap_{\zeta<\xi} A_0^\zeta.$$

Thereby we get $\langle r_{\xi,\eta} \mid \xi < \delta_0, \eta < \delta_1\rangle$ and $\langle \bar{u}_1(\xi) \mid \xi < \delta_0\rangle$. Let us show how one stabilizes the first two measure one sets of $r_{\xi,\eta}$, $\langle A_0^{\xi,\eta}, A_1^{\xi,\eta}\rangle$. [28] This

[28] Again, assume that $\bar{n} = 0$. In the general case we would be stabilizing measure one sets at coordinates $i \leq \bar{n}+1$. The argument is the same.

will enable us to take lower bounds upon the $\bar{u}_1(\xi)$'s. For each $\xi < \delta_0$ let $\Phi_\xi \colon \eta \mapsto \langle A_0^{\xi,\eta}, A_1^{\xi,\eta}\rangle$. Since δ_1 is measurable and $2^{\kappa_1} < \delta_1$ we find $B_{1,\xi} \in \mathcal{U}_1$ where Φ_ξ is constant; say with value $\langle A_0^\xi, A_1^\xi\rangle$. Since $\delta_0 < \kappa_1$ it follows that $B_1 := \bigcap_{\xi<\delta_0} B_{1,\xi} \in \mathcal{U}_1$ and $A_1 := \bigcap_{\xi<\delta_0} A_1^\xi$ is $E_{1,\mathrm{mc}(a_1)}$-large, where a_1 is common among the $r_{\xi,\eta}$'s (by construction). Finally, use that $2^{\kappa_0} < \delta_0$ to find $B_0 \in \mathcal{U}_0$ for which A_0^ξ is constant; say with value A_0. Let us now look at the sequences $\langle r_{\xi,\eta} \mid \xi \in B_0,\, \eta \in B_1\rangle$ and $\langle \bar{u}_1(\xi) \mid \xi \in B_0\rangle$. By construction,

$$\bar{u}_1(\xi)^{\frown +1} \leq \pi(r_{\xi,\eta})^{\frown +1} \text{ for all } \eta < \delta_1,$$

so that the first two measure one sets of $\bar{u}_1(\xi)$ are $\pi_{\mathrm{mc}(a_0),\mathrm{mc}(a_0\cap\lambda)}$ "A_i^ξ. Shrink the second measure one set of each $r_{\xi,\eta}$ (i.e., A_1^ξ) to A_1. Do the same with $\bar{u}_1(\xi)$ using $\pi_{\mathrm{mc}(a_1),\mathrm{mc}(a_1\cap\lambda)}$ "A_1. Keep denoting the resulting conditions by $r_{\xi,\eta}$ and $\bar{u}_1(\xi)$. Note that now we are in conditions of taking a $\leq^{*,1}$-lower bound upon the $\langle \bar{u}_1(\xi) \mid \xi \in B_0\rangle$; call this $\bar{u}_1$. Finally, let $y_1 := \bigcup_{\xi\in B_0,\eta\in B_1} y_{\xi,\eta}$ where $y_{\xi,\eta} \in \mathcal{P}_\lambda(\theta)$ are given by the Splitting Lemma.

Recalling items (1)–(5) at the beginning of the proof (page 40) one confirms that $\langle r_{\xi,\eta} \mid \xi \in B_0,\, \eta \in B_1\rangle$ and $\bar{u}_1$ are as wished. In general one proceeds by induction as explained above, allowing the B_i's to get shrunk. Since the measures $\mathcal{U}_n$ are countably complete, we are fine in taking intersections of these sets at the end; these will be the final Y_n's supporting the tree of $r_{\vec{\sigma}}$'s.

Let us now complete the proof of ITP(λ). As of now we have defined

$$\langle r_{\vec{\sigma}} \mid \vec{\sigma} \in \textstyle\prod_{n<m} Y_n,\, n < \omega\rangle,\ \langle \bar{u}_1 \mid n < \omega\rangle \text{ and } \langle y_n \mid n < \omega\rangle,$$

in the ground model, V. Let $\bar{u}_\omega$ be a $\leq^*$-lower bound for $\langle \bar{u}_n \mid n < \omega\rangle$ and let $G \subseteq \mathbb{P}_\lambda$ a V-generic with $\bar{u}_\omega \in G$.[29]

Claim 6.10.2. ITP(λ) *holds in* $V[G]$.

Proof of claim. Since $\dot{b}$ was forced to be an ineffable branch which is $<\lambda$-approximated in $V[G]$, letting $c \in \mathcal{P}_\lambda(\theta)^{V[G]}$ with $c \supseteq \bigcup_{n<\omega} y_n$ we infer that $b \cap c \in V[G]$. Inside $V[G]$ the product $\prod_{n<\omega} \delta_n$ has cardinality λ.

For each $f \in \prod_{n<\omega} Y_n$ let r_f be a $\leq^*$-lower bound for $\langle r_{f\restriction n} \mid n < \omega\rangle$. Such a condition exists because the G-part of each $r_{f\restriction n}$ is captured by $\bar{u}_\omega$. By construction, every $\bar{n}$-extension of r_f decides the value of $\dot{b} \cap c$. Let s_f be one of such extensions and let $b_f \in V[G]$ be the decided value. Since $\mathbb{P}_\lambda$ is λ-cc and $\dot{b} \cap c$ has size $<\lambda$, there are $<\lambda$-many possible b_f's. Working inside $V[G]$, consider the map $\Phi \colon f \mapsto b_f$. Using Clause (5) in page 40 one can show that Φ is one-to-one. Therefore the are at least λ-many such b_f's. This produces the desired contradiction and establishes ITP(λ) in $V[G]$. □

The proof of the lemma has been completed. □

[29]Note that the arguments at the beginning of this section (see p.33) hold for an arbitrary V-generic filter and thus they remain valid for our particular G with $\bar{u}_\omega \in G$.

7. The proof of the main theorem

Assume the GCH holds and that there is an increasing sequence of supercompact cardinals $\langle\kappa_n \mid n<\omega\rangle^\frown\langle\lambda\rangle$ with $\lambda > \sup_{n<\omega}\kappa_n$. By preparing the universe we may assume that each κ_n is indestructible under κ_n-directed-closed forcing that preserve the GCH. Let $\langle\delta_n \mid n<\omega\rangle$ be measurable cardinals with $\kappa_n < \delta_n < \kappa_{n+1}$. Forcing with the full-supported product $\prod_{n<\omega}\mathrm{Col}(\kappa_n^{++}, {<}\delta_n)$ makes each δ_n become κ_n^{+3} and yields a precipitous ideal $\mathcal{I}_n$ (over δ_n) whose positive sets $\mathcal{I}_n^+$ forms a κ_n^{++}-strategically closed forcing. This generic extension will be our ground model, which we call V.

In this section we assume that $\mathbb{P}_\lambda$ is the iteration with collapses (see §5). More explicitly, $\mathbb{P}_\lambda$ is the iteration starting with the Mitchell EBPF with collapses (Definition 3.11) which at successors stages invokes the functor $\mathbb{A}(\cdot,\cdot)$ of §4. We bear on all the tree-property-related considerations described at the beginning of §6. Recall that the EBPFC carries a fusion order $\leq^{*,k}$ (see Definition 3.5) which is κ_k^{+4}-closed. This induces a corresponding fusion order $\leq^{*,k}$ over $j(\mathbb{P}_\lambda)$ in a natural way (see Definition 6.2).

Definition 7.1. Let $u, v \in j(\mathbb{P}_\lambda)$ be with the same length ℓ and $k \geq \ell$.

Write $u^{\frown +k} \leq v^{\frown +k}$ if $\mathsf{EBPF}(u) \leq^{*,k} \mathsf{EBPF}(v)$ holds but the requirement $\mathsf{EBPF}(u) \leq^* \mathsf{EBPF}(v)$ is replaced by

$$u^\frown\vec{\nu} \leq^* v^\frown\vec{\nu} \text{ for all } \vec{\nu} \in \textstyle\prod_{\ell\leq i\leq k} A_i.$$

Almost verbatim the proof of Lemma 6.6 yields:

Lemma 7.2.

1. *$\langle j(\mathring{\mathbb{P}}_\lambda), {}^{\frown +k}\rangle$ is κ_k^{+4}-closed.*
2. *If $u^{\frown +k} \leq v^{\frown +k}$ then there is $w \leq^{*,k} v$ such that $w^{\frown +k} \leq u^{\frown +k}$.* □

The *Splitting Lemma* in the collapsing scenario is phrased the same way as before but there are some worth-mentioning differences in its proof.

We run the proof for a fixed k recalling that $\delta_k = \kappa_k^{+3}$.

Proof of the Splitting Lemma with collapses. Let us define $\langle (u_\xi, v_\xi) \mid \xi < \delta\rangle$ and $\langle \mathcal{M}_\xi \mid \xi < \delta\rangle$ be as in the beginning of Lemma 6.8. We just describe the construction of the u_ξ's and v_ξ's; the $\mathcal{M}_\xi$'s are chosen as before.

Suppose that $\langle (u_\xi, v_\xi) \mid \xi < \zeta\rangle$ has been defined. Let v be a $\leq^{*,k}$-lower bound for the previous v_ξ's and $u_\zeta \leq^{*,k,-} v$ be in $j(\mathring{\mathbb{P}}_\lambda)$ all of whose $\bar{n}$-extensions decide $\dot{b} \cap y_\zeta$ (by Lemma 6.3 and Footnote 24). We define v_ζ.

<u>EBPF-part:</u> The EBPF-part of v_ζ, q_ζ, is defined as follows:

$$\langle (f_0^\zeta, \rho_0, \vec{h}_0), \ldots, (a_\ell, A_\ell, f_\ell^\zeta, \vec{F}_\ell), \ldots, (a_k, A_k, f_k^\zeta, \vec{F}_k), (p_\zeta)_{k+1}, (p_\zeta)_{k+2}, \ldots\rangle,$$

where ρ_i's $\vec{h}_i$'s and (a_i, A_i)'s are from the original condition.

Observe that $q_\zeta \leq^{*,k} \mathsf{EBPF}(v)$.

Mitchell-part: Let d_ζ be the function with $\mathrm{dom}(d_\zeta) := \mathrm{dom}(c_\zeta)$ which for each $\alpha \in \mathrm{dom}(d_\zeta)$ the domain of $d_\zeta(\alpha)$ equals

$$\{\langle \vec{\pi}, t\rangle \mid \vec{\pi} \in \textstyle\prod_{\ell \leq i \leq m} \pi_{\mathrm{mc}(a_i^{q_\zeta}),\mathrm{mc}(a_i^{q_\zeta}\cap\alpha)} \text{“} A_i^{q_\zeta} \wedge t \in \bigcup_{n\leq m}(\mathbb{S}_n \downarrow \varpi_n(q_\zeta \restriction \alpha^\frown \vec{\pi}))\}.$$

Call $\langle \vec{\pi}, t\rangle$ *good* if $|t| \geq k$ (hence, $|\vec{\pi}| \geq k - \ell$) and $\langle \vec{\pi}, t\rangle \in \mathrm{dom}(c_\zeta(\alpha))$.

If $\langle \vec{\pi}, t\rangle$ is good, then for all $\alpha \in \mathrm{dom}(d_\zeta)$, define

$$d_\zeta(\alpha, \vec{\pi}, t) = c_\zeta(\alpha, \vec{\pi}, t).$$

Otherwise, split the definition of $d_\zeta(\alpha, \vec{\pi}, t)$ into two cases:

$$\begin{cases} d(\alpha, \langle \pi_{\mathrm{mc}(a_i^{q_\zeta}\cap\alpha),\mathrm{mc}(a_i^{q}\cap\alpha)}(\pi_i) \mid \ell \leq i \leq t\rangle, \mathbf{Col}(q_\zeta \restriction \alpha, \vec{\pi}, t)), & \text{if } \alpha \in \mathrm{dom}(d); \\ \{\langle \check{\varnothing}, \mathbb{1}_{\mathbb{P}\restriction\alpha}\rangle\}, & \text{otherwise.} \end{cases}$$

By definition of q_ζ, $(q_\zeta \restriction \alpha^\frown \vec{\pi}) + t = (p_\zeta \restriction \alpha^\frown \vec{\pi}) + t$ for good pairs $\langle \vec{\pi}, t\rangle$. The point is that q_ζ and p_ζ differ only on the collapsing parts, $\vec{h}$ and $\vec{F}$. This implies that d_ζ is a well-defined Mitchell-strategy.

Sharon-part: This is defined exactly as in the proof without collapses. The difference is that now we obtain $v_\xi \leq^{*,k} v$ such that for $\vec{\nu} \in \prod_{\ell \leq i \leq k} A_i^{u_\xi}$,

$$(v_\xi{}^\frown \vec{\nu}) + \varpi_k(u_\xi{}^\frown \vec{\nu}) \leq^* u_\xi{}^\frown \vec{\nu}.$$

Namely, for the desired inequality to be true we need to add to $v_\xi{}^\frown \vec{\nu}$ the collapsing part of $u_\xi{}^\frown \vec{\nu}$ appearing in coordinates $[\ell, k]$.

Thus we get $\langle (u_\xi, v_\xi) \mid \xi < \delta\rangle$ and $\langle \mathcal{M}_\xi \mid 0 < \xi < \delta\rangle$. As before, find an unbounded set $I \subseteq \delta$ of indices ξ's such that $\langle A_\ell^\xi, \dots, A_k^\xi\rangle$ is constant; say, with value $\langle B_\ell, \dots, B_k\rangle$. Similarly, $\langle \vec{h}_0^\xi, \dots, \vec{h}_{\ell-1}^\xi\rangle$ and $\langle \vec{F}_\ell^\xi, \dots, \vec{F}_k^\xi\rangle$ can be stabilized to certain values $\langle \vec{g}_0, \dots, \vec{g}_{\ell-1}\rangle$ and $\langle \vec{G}_\ell, \dots, \vec{G}_k\rangle$.[30]

Let u_δ be a $\leq^{*,k}$-lower bound for the v_ξ's and strengthen it so that its first $(k-\ell)$-many measure one sets are $\langle B_\ell, \dots, B_k\rangle$, its $\vec{h}$-part is $\langle \vec{g}_0, \dots, \vec{g}_{\ell-1}\rangle$ and its $\vec{F}$-part, $\langle \vec{G}_\ell, \dots, \vec{G}_k\rangle$. Note that for all $\vec{\nu} \in \prod_{\ell \leq i \leq k} B_i$ and $\xi \in I$,

$$u_\delta{}^\frown \vec{\nu} \leq^* (v_\xi{}^\frown \vec{\nu}) + \varpi_k(u_\xi{}^\frown \vec{\nu}) \leq^* u_\xi{}^\frown \vec{\nu}.$$

In fact $\langle u_\xi{}^\frown \vec{\nu} \mid \xi \in I\rangle$ is decreasing for each such $\vec{\nu}$. So, $u_\delta{}^{\frown +k} \leq u_\xi{}^{\frown +k}$ for all $\xi \in I$. Re-enumerate $\langle u_\xi \mid \xi \in I\rangle$ by $\langle u_\xi \mid \xi < \delta\rangle$.

From this point on the proof is exactly the same as in Lemma 6.8 using the new version of Lemma 6.6 (namely, Lemma 7.2). $\square$

Using the *Splitting Lemma* and Lemma 7.2 a similar argument to that given in Theorem 6.10 yields $\mathrm{ITP}(\lambda)$ in a generic extension by $\mathbb{P}_\lambda$. Note that, while now the δ_n's are no longer measurable, they carry precipitous ideals $\mathcal{I}_n$ whose positive sets $\mathcal{I}_n^+$ are κ_n^{+2}-strategically closed. This suffices to ensure in page 42 that $\bigcap_{\xi<\delta_0} B_{1,\xi} \in \mathcal{I}_1^+$ and $B_0 \in \mathcal{I}_0^+$ (note that $2^{\kappa_0} < \kappa_0^{+2} < \delta_0 < \delta_1$)

In the end we obtain the following:

[30] See our comments in **Reason 2**, p. 18.

Main Theorem. *Assume the* GCH *holds. Suppose that* $\langle \kappa_n \mid n < \omega\rangle^\frown\langle\lambda\rangle$ *is an increasing sequence of supercompact with* $\lambda > \sup_{n<\omega}\kappa_n$. *Then there is a generic extension of the set-theoretic universe where the following hold:*

(1) $\aleph_\omega = \sup_{n<\omega}\kappa_n$ *is strong limit and* $\lambda = \aleph_{\omega+2}$*;*
(2) $\mathsf{GCH}_{<\aleph_\omega}$ *holds while* $2^{\aleph_\omega} = \aleph_{\omega+2}$*;*
(3) $\mathrm{Refl}(\aleph_{\omega+1})$ *holds;*
(4) $\mathrm{ITP}(\aleph_{\omega+2})$ *holds.*

Proof. Let $\mathbb{P}_\lambda$ be the $(\Sigma, \vec{\mathbb{S}})$-Prikry-styled iteration from the beginning of the section. Appealing to the arguments in [PRS21, Theorem 8.8] we have that in a generic extension by $\mathbb{P}_\lambda$ the following hold: $\aleph_\omega$ is strong limit, $\mathsf{GCH}_{<\aleph_\omega}$, $2^{\aleph_\omega} = \aleph_{\omega+2}$ and $\mathrm{Refl}(\aleph_{\omega+1})$. Note that for the latter property we need the usual *catch-your-tail argument* which works in that $\mathbb{P}_\lambda$ is a λ-cc iteration with supports of size $\leq\kappa$. Finally $\mathrm{ITP}(\aleph_{\omega+2})$ holds in the purported generic extension by virtue of the preceding arguments. □

8. Acknowledgments

Poveda acknowledges the support of the Center of Mathematical Sciences and Applications at Harvard University. Sinapova was partially supported by the National Science Foundation under Grant No. DMS - 1954117.

References

[BE51] NG de Bruijn and P Erdos. A colour problem for infinite graphs and a problem in the theory of relations. *Indigationes Mathematicae*, 13:371–373, 1951.

[CF98] James Cummings and Matthew Foreman. The tree property. *Advances in Mathematics*, 133(1):1–32, 1998.

[Git10] Moti Gitik. Prikry-type forcings. In *Handbook of set theory. Vols. 1, 2, 3*, pages 1351–1447. Springer, Dordrecht, 2010.

[Git11] Moti Gitik. Extender based forcings, fresh sets and Aronszajn trees. 2011. preprint.

[Git19] Moti Gitik. Collapsing generators. *Preprint*, 2019.

[Git22] Moti Gitik. Reflection and not SCH with overlapping extenders. *Archive for Mathematical Logic*, 61(5):591–597, 2022.

[GM94] Moti Gitik and Menachem Magidor. Extender based forcings. *J. Symbolic Logic*, 59(2):445–460, 1994.

[HS19] Sherwood Hachtman and Dima Sinapova. ITP, ISP, and SCH. *The Journal of Symbolic Logic*, 84(2):713–725, 2019.

[HU20] Yair Hayut and Spencer Unger. Stationary reflection. *The Journal of Symbolic Logic*, 85(3):937–959, 2020.

[Jec73] Thomas J Jech. Some combinatorial problems concerning uncountable cardinals. *Annals of Mathematical Logic*, 5(3):165–198, 1973.

[Mag74] Menachem Magidor. Combinatorial characterization of supercompact cardinals. *Proceedings of the American Mathematical Society*, 42(1):279–285, 1974.

[Mag77] Menachem Magidor. On the singular cardinals problem. II. *Ann. of Math. (2)*, 106(3):517–547, 1977.

[Mag82] Menachem Magidor. Reflecting stationary sets1. *The Journal of Symbolic Logic*, 47(4):755–771, 1982.

[Mit72] William Mitchell. Aronszajn trees and the independence of the transfer property. *Annals of Mathematical Logic*, 5(1):21–46, 1972.

[Pov20] Alejandro Poveda. *Contributions to the theory of Large Cardinals through the method of Forcing*. 2020. Thesis (Ph.D.)– Universitat de Barcelona.

[PRS19] Alejandro Poveda, Assaf Rinot, and Dima Sinapova. Sigma-Prikry forcing I: The Axioms. *Canadian Journal of Mathematics*, pages 1–38, 2019.

[PRS21] Alejandro Poveda, Assaf Rinot, and Dima Sinapova. Sigma-Prikry forcing III: Down to $\aleph_\omega$. *Submitted*, 2021.

[PRS22] Alejandro Poveda, Assaf Rinot, and Dima Sinapova. Sigma-prikry forcing II: iteration scheme. *Journal of Mathematical Logic*, 22(03):2150019, 2022.

[Spe90] Ernst Specker. Sur un probleme de sikorski. *Ernst Specker Selecta*, pages 49–52, 1990.

[Wei12] Christoph Weiß. The combinatorial essence of supercompactness. *Annals of Pure and Applied Logic*, 163(11):1710–1717, 2012.

Center of Mathematical Sciences and Applications, Harvard University, Cambridge, MA 02138, USA

Email address: alejandro@cmsa.fas.harvard.edu

URL: https://scholar.harvard.edu/apoveda/home

Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, USA.

Email address: dima.sinapova@rutgers.edu

URL: https://sites.math.rutgers.edu/~ds2005/